I NUMERI COMPLESSI

- $\alpha = -1 + i \qquad \beta = 1 - 2i \qquad \gamma = -i$

$z_1 = \alpha + \beta = -1 + i + 1 - 2i = -i$

$z_2 = \alpha \cdot \beta = (-1+i)(1-2i) = -1 + i + 2i + 2 = 1 + 3i$

$$z_3 = \frac{\alpha - \beta}{\alpha^2 + \gamma^2} = \frac{-1+i-1+2i}{(-1+i)^2 + (-i)^2} = \frac{-2+3i}{\cancel{1}-1-2i-1} = \frac{-2+3i}{-1-2i} \cdot \frac{-1+2i}{-1+2i} =$$

$$= \frac{2-3i-4i-6}{1+4} = \frac{-4-7i}{5} = -\frac{4}{5} - \frac{7}{5}i$$

- $|z - i| = |z+2|$ trovare lo z immaginario puro

$z = iy = -\frac{3}{2}i$

$|iy - i| = |iy + 2|$

$|i(y-1)| = |iy + 2|$

$\sqrt{0^2 + (y-1)^2} = \sqrt{2^2 + y^2}$

$\cancel{y^2} + 1 - 2y = 4 + \cancel{y^2} \qquad y = -\frac{3}{2}$

! $\sqrt[8]{z^6} \neq \sqrt[4]{z^3}$!

PERDO 4 SOLUZIONI IN $\mathbb{C}$

- Calcolare $\sqrt[5]{1-i}$

$w_k = \sqrt[5]{1-i} \quad k = 0 \ldots 4$

$z = 1 - i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

$$\begin{cases} |w_k| = \sqrt[5]{\sqrt{2}} = \sqrt[10]{2} \\ \varphi_k = \dfrac{-\frac{\pi}{4} + 2k\pi}{5} \end{cases}$$

$\varphi_0 = -\frac{\pi}{20} \qquad w_0 = \sqrt[10]{2} \cdot e^{-i\frac{\pi}{20}}$

$\varphi_1 = \frac{7}{20}\pi$

$\varphi_2 = \frac{3}{20}\pi$

$\varphi_3 = \frac{23}{20}\pi$

$\varphi_4 = \frac{31}{20}\pi$

• $z^4-(1+i)z^2+i=0$

$$z^2=\frac{1+i\pm\sqrt{1+i^2+2i-4i}}{2}=\frac{(1+i)\pm\sqrt{(1-i)^2}}{2}=\frac{(1+i)\pm(1-i)}{2}\ \begin{cases}1\\ i\end{cases}$$

$$z^2=\frac{1+i\pm\sqrt{-2i}}{2}$$

$$-2i=2e^{-i\frac{\pi}{2}}$$

$$\sqrt{-2i}=\sqrt{2}\cdot e^{i\left(\frac{-\frac{\pi}{2}+2k\pi}{2}\right)}\ \begin{cases}\sqrt{2}\,e^{i\left(-\frac{\pi}{4}\right)}=1-i\\ \sqrt{2}\,e^{i\left(-\frac{\pi}{4}+\pi\right)}=-1+i\end{cases}$$

....

$$z^2=1\implies z=\pm\sqrt{1}$$

$$z^2=i\implies z=\pm\sqrt{i}$$

$$\sqrt{i}=\sqrt{1}\cdot e^{i\left(\frac{\frac{\pi}{2}+2k\pi}{2}\right)}\quad \text{con } k=0,1$$

$$e^{i\frac{\pi}{4}}=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$$

$$e^{i\left(\frac{\pi}{4}+\pi\right)}=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

$$\boxed{\begin{aligned}z_1&=+1+i0\\ z_2&=-1+i0\\ z_3&=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\\ z_4&=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\end{aligned}}$$

• $z^4=8\bar{z}$

$$z=re^{i\varphi}\quad e\quad \bar{z}=re^{-i\varphi}$$

$$r^4e^{4i\varphi}=8re^{-i\varphi}$$

$$r(r^3e^{4i\varphi}-8e^{-i\varphi})=0\qquad r=0$$

$$r^3e^{4i\varphi}=8e^{-i\varphi}\implies\begin{cases}r^3=8 & r=2\\ 4\varphi=-\varphi+2k\pi & k\in\mathbb{Z}\end{cases}$$

$$5\varphi=2k\pi\qquad \varphi=\frac{2}{5}k\pi$$

$$z_k=2e^{i\frac{2}{5}k\pi}$$

Dopo 5 volte si ripete

$$z=0+i0$$

$$z_0=2e^{i0}=2$$

$$z_1=2e^{i\frac{2}{5}\pi}=2\left(\cos\tfrac{2}{5}\pi+i\sin\tfrac{2}{5}\pi\right)$$

$$z_2=2e^{i\frac{4}{5}\pi}=\ldots$$

$$z_3=2e^{i\frac{6}{5}\pi}=\ldots$$

$$z_4=2e^{i\frac{8}{5}\pi}=\ldots$$

FUNZIONI

$$\boxed{f: A \to B \quad x \to y=f(x) \qquad A,B \subseteq \mathbb{R}}$$

- $f_1 : \mathbb{R} \to \mathbb{R}$, $x \to y = x^2$ $\qquad$ • $f_2 : [0,1) \to \mathbb{R}$, $x \to y = x^2$

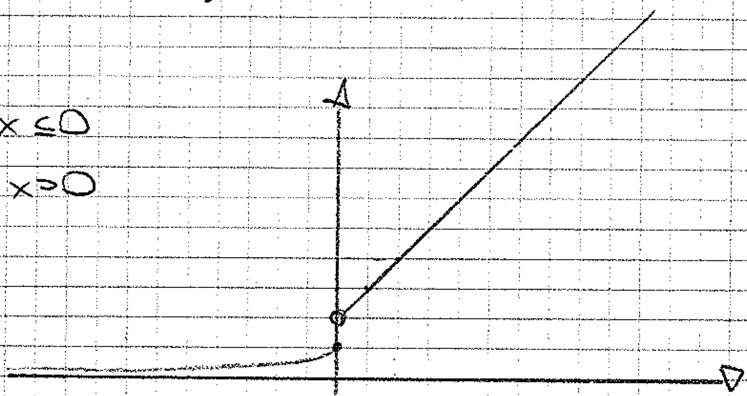
- $f : \mathbb{R} \to \mathbb{R}$, $x \to y = 2^x$ $\qquad f|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}$, $x \to y = 2^x$ $\qquad$ RESTRIZIONE DELLA FUNZIONE

CAMPO DI ESISTENZA = PIÙ GRANDE DOMINIO POSSIBILE

- $z : \mathbb{R}^+ \to \mathbb{R}$, $x \to y = \sqrt{x}$ $\qquad$ di z, da $\mathbb{R}^+$ a valori in $\mathbb{R}$ che a x associa $y = \sqrt{x} \in \mathbb{R}$

$\qquad C.E: \mathbb{R}^+ \cup \{0\}$

$$y = \begin{cases} 2^x & x \le 0 \\ x+2 & x > 0 \end{cases}$$



- Se $A = B = \mathbb{R}$ $\qquad f([0,1)) = \{1\} \cup (2,3)$

$f(\mathbb{R}^-) = (0,1)$

$f([-1,1]) = [\frac{1}{2}, 1] \cup (2,3]$

$f(A) = f(\mathbb{R}) = (0,1] \cup (2,+\infty)$

$f(\{3\}) = \{5\}$

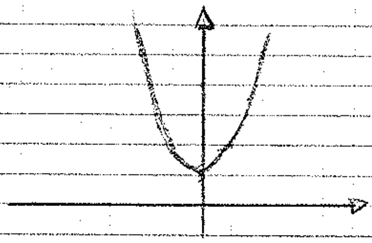
Se $A = \mathbb{R}$ e $B = (0,1] \cup (2,+\infty)$

$f(A) = B$ è suriettiva

Se $A = B = \mathbb{R}$ $\qquad f^{-1}(\{5\}) = \{3\}$

$f^{-1}((0,4)) = (-\infty, 2)$

$f(x) : \mathbb{R} \to \mathbb{R}$, $x \to y = x^2+1$



$f^{-1}(\{5\}) = \{-2\} \cup \{2\}$

$f(\mathbb{R}) = [1, +\infty)$ non è suriettiva

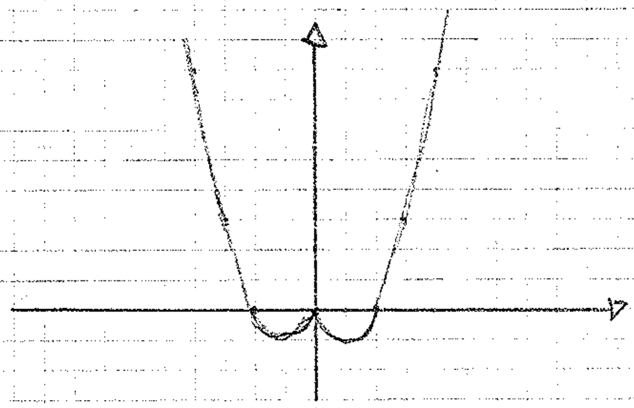
$\tilde{f} : (-\infty, 0] \to [1, +\infty)$, $x \to y = x^2+1$ $\qquad \to$ BIUNIVOCA (sia iniettiva che suriettiva) BIETTIVA

INVERTIBILITÀ

$$\boxed{\text{Se } f: A \to B,\ x \to f(x)=y \text{ è iniettiva e suriettiva} \Rightarrow f^{-1}: B \to A,\ y = x \,/\, f(x)=y}$$

$\times$ $f: \mathbb{R} \to \mathbb{R}$
$x \to y = x^2 - 2|x|$

$$y = x^2 - 2|x| = \begin{cases} x^2 - 2x & x \geq 0 \\ x^2 + 2x & x < 0 \end{cases}$$



SURIETTIVITÀ

$\bar{f}: \mathbb{R} \to [-1, +\infty)$
$x \to y = x^2 - 2|x|$

INIETTIVITÀ

$$\left.\begin{array}{l} \bar{f}_1: (-\infty, -1] \to [-1, +\infty) \\ \bar{f}_2: [-1, 0] \to [-1, 0] \\ \bar{f}_3: [0, 1] \to [-1, 0] \\ \bar{f}_4: [1, +\infty) \to [-1, +\infty) \end{array}\right\} \text{4 restrizioni biettive}$$

$x^2 + 2x = y$
$x^2 + 2x - y = 0$
$x = -1 \pm \sqrt{y+1}$

$f_1^{-1}: [-1, +\infty) \to (-\infty, -1]$
$x \to y = -1 - \sqrt{x+1}$

$f_2^{-1}: [-1, 0] \to [-1, 0]$
$x \to y = -1 + \sqrt{x+1}$

$f_3^{-1}: [-1, 0] \to [0, 1]$
$x \to y = 1 - \sqrt{x+1}$

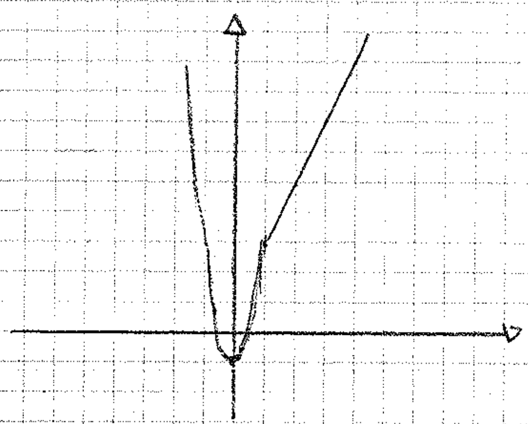
$f_4^{-1}: [-1, +\infty] \to [1, +\infty)$
$x \to y = 1 + \sqrt{x+1}$

> LA FUNZIONE INVERSA È IL SIMMETRICO RISPETTO ALLA BISETTRICE DEL 1°/3° QUADRANTE $y=x$

× $y=(2x+1)(x-|x-1|)=\begin{cases}2x+1 & x>1\\ 4x^2-1 & x<1\end{cases}$

$f_{I}:(-\infty,0]\to[-1,+\infty)$
$x\to y=4x^2-1$

$f_{II}:[0,+\infty)\to[-1,+\infty)$
$x\to y=\begin{cases}2x+1 & x\geq 1\\ 4x^2-1 & 0\leq x<1\end{cases}$



$y=4x^2-1 \qquad x=\pm\frac{1}{2}\sqrt{y+1}$
$y=2x+1 \qquad x=\frac{1}{2}(y-1)$

$\Downarrow$

$f_{I}^{-1}:[-1,+\infty)\to(-\infty,0]$
$x\to y=-\frac{1}{2}\sqrt{x+1}$

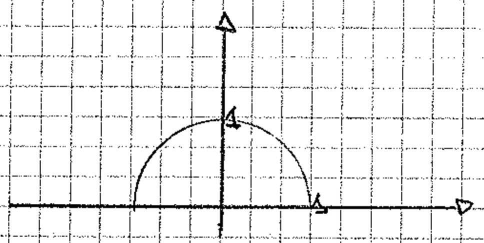
$f_{II}^{-1}:[-1,+\infty)\to[0,+\infty)$
$x\to y=\begin{cases}\frac{1}{2}(x-1) & x\geq 3\\ \frac{1}{2}\sqrt{x+1} & -1\leq x<3\end{cases}$

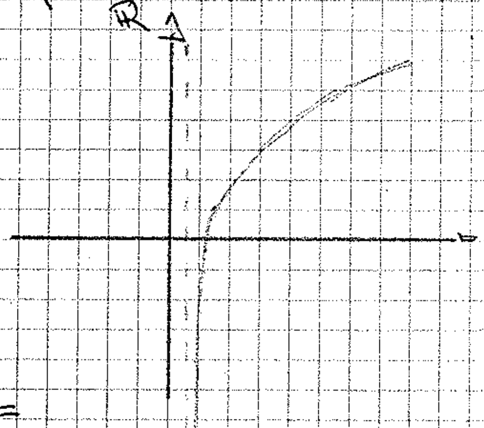
FUNZIONE COMPOSTA

$g[f(x)]=g\circ f$

- $\operatorname{Im} f\cap \operatorname{dom} g\neq\emptyset$
- $\operatorname{dom} g\circ f=f^{-1}\{\operatorname{Im} f\cap \operatorname{dom} g\}$
- $\operatorname{Im} g\circ f=g\{\operatorname{Im} f\cap \operatorname{dom} g\}$

× $f(x)=\sqrt{1-x^2} \qquad f:[-1,1]\to[0,1]$
$g(x)=\ln(x-\frac{1}{2}) \qquad g:(\frac{1}{2},+\infty)\to(-\infty,+\infty)$





$-\operatorname{Im} f\cap \operatorname{dom} g=[0,1]\cap(\frac{1}{2},+\infty)=$
$=(\frac{1}{2},1]\neq\emptyset$

$\operatorname{dom} g\circ f=f^{-1}\{(\frac{1}{2},1]\}=(-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2})$

$\operatorname{Im} g\circ f=g\{(\frac{1}{2},1]\}=(-\infty,-\ln 2)$

$\Rightarrow g\circ f:(-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2})\to(-\infty,-\ln 2]$
$x\to y=\ln(\sqrt{1-x^2}-\frac{1}{2})$

$- \; \mathrm{Im}\, g \cap \mathrm{dom}\, f = \mathbb{R} \cap [-1,1] = [-1,1] \neq \emptyset$

$$\mathrm{dom}\, f\circ g = g^{-1}\{[-1,1]\} = \left[\frac{1}{2}+\frac{1}{e}, \frac{1}{2}+e\right]$$

$$\mathrm{Im}\, f\circ g = f\{[-1,1]\} = [0,1]$$

$$f\circ g : \left[\frac{1}{2}+\frac{1}{e}, \frac{1}{2}+e\right] \to [0,1]$$

$$x \to y = \sqrt{1-\ln^2\left(x-\frac{1}{2}\right)}$$

$\times$ $f(x) = 4 - x^2 \qquad f: \mathbb{R} \to (-\infty, 4]$

$g(x) = 2^x \qquad g: \mathbb{R} \to \mathbb{R}^+$

$- \; \mathrm{Im}\, f \cap \mathrm{dom}\, g = (-\infty, 4] \cap \mathbb{R} = (-\infty, 4]$

$$\mathrm{dom}\, g\circ f = f^{-1}\{(-\infty, 4]\} = \mathbb{R}$$

$$\mathrm{Im}\, g\circ f = g\{(-\infty, 4]\} = (0, 16]$$

$$g\circ f : \mathbb{R} \to (0, 16]$$

$$x \to y = 2^{4-x^2}$$

g sempre crescente

f $\left\langle \begin{array}{l} \text{crescente su } (-\infty, 0] \\ \text{decrescente su } [0, +\infty) \end{array}\right. \Rightarrow g\circ f \left\langle \begin{array}{l} \text{crescente su } (-\infty, 0] \\ \text{decrescente su } [0, +\infty) \end{array}\right.$

$$g\circ f_1 : (-\infty, 0] \to (0, 16]$$

$$x \to y = 2^{4-x^2}$$

$$(g\circ f_1)^{-1} = (f^{-1}\circ g^{-1})_1 : (0, 16] \to (-\infty, 0]$$

$$x \to y = -\sqrt{4 - \log_2 x}$$

$\times$ $f(x) = \dfrac{x+1}{x-2} \qquad f: (-\infty, 2)\cup(2, +\infty) \to (-\infty, 1)\cup(1, +\infty)$

$g(x) = \mathrm{arctg}\, x \qquad g: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$- \; \mathrm{Im}\, f \cap \mathrm{dom}\, g = \mathrm{Im}\left[(-\infty, 1)\cup(1, +\infty)\right] \cap \mathbb{R} = \mathrm{Im}\, f \neq \emptyset$

$$\mathrm{dom}\, g\circ f = f^{-1}(\mathrm{Im}\, f) = \mathrm{dom}\, f$$

$$\mathrm{Im}\, g\circ f = g\left((-\infty, 1)\cup(1, +\infty)\right) = \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)\cup\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\hookrightarrow g\circ f : (-\infty, 2)\cup(2,+\infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)\cup\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$x \to y = \operatorname{arctg}\frac{x+1}{x-2}$$

I LIMITI

LIMITI FONDAMENTALI

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2} \qquad \lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \alpha \in \mathbb{R}$$

$$*\lim_{x\to 0}\frac{\operatorname{tg} x}{x} = 1 \qquad \lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$*\lim_{x\to 0}\frac{\arcsin x}{x} = 1 \qquad \lim_{x\to +\infty}\frac{e^x}{x^k} = +\infty \quad \forall k$$

$$*\lim_{x\to 0}\frac{\operatorname{arctg} x}{x} = 1 \qquad \lim_{x\to 0^+} x^k \cdot \ln x = 0 \quad k>0$$

$$\lim_{x\to 0}\frac{e^x - 1}{x} = 1 \qquad \lim_{x\to +\infty}\frac{\ln x}{x^k} = 0 \quad k>0$$

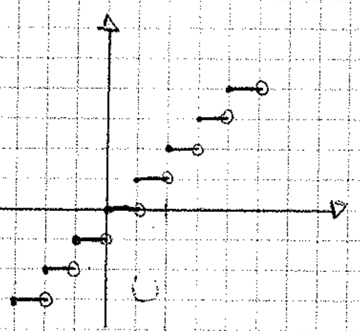
PARTE INTERA

$y = [x] = \operatorname{int}(x) = E(x)$

$[x] : \mathbb{R} \to \mathbb{Z}$

$x \to \ \in \mathbb{Z}$

con α il più grande intero relativo che non supera x



es. $[4.3] = 4$ $\quad [5] = 5$

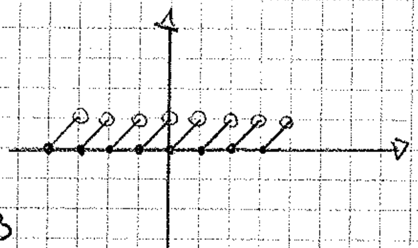
$[0] = 0$ $\quad [-2.3] = -3$

$$(x-1) < [x] \le x$$

MANTISSA

$y = \operatorname{Mant}(x) = M(x) = x - [x]$

$M(x) : \mathbb{R} \to [0,1)$



es. $M(3.2) = 3.2 - 3 = 0.2$

$M(3) = 0$ $\quad M(7.763) = 0.763$

$M(-3.4) = -3.4 + 4 = 0.6$ $\quad M(7.\bar{9}) = M(8) = 0$

$M(-81.9) = 0.1$ $\quad M(81.9) = 0.9$

$$0 \le M(x) < 1$$

TEOREMA DEL CONFRONTO

- $\lim_{x\to+\infty} \frac{\sqrt{5+\cos x}}{x^2+1} = 0$

$-1 \le \cos x \le 1 \qquad \forall x \in \mathbb{R}$

$4 \le 5+\cos x \le 6 \qquad \sqrt{x}$ continua e crescente

$\sqrt{4} \le \sqrt{5+\cos x} \le \sqrt{6}$

$$\underbrace{\frac{\sqrt{4}}{x^2+1}}_{\downarrow\, 0} \le \underbrace{\frac{\sqrt{5+\cos x}}{x^2+1}}_{\Downarrow\, T.C.\ 0} \le \underbrace{\frac{\sqrt{6}}{x^2+1}}_{\downarrow\, 0}$$

- $\lim_{x\to+\infty} \frac{\sin x \cos x}{x+3} = \lim_{x\to+\infty} \frac{\frac{1}{2}\sin 2x}{x+3} = 0$

$\sin 2x = 2\sin x\cos x$

$-\frac{1}{2} \le \frac{1}{2}\sin 2x \le \frac{1}{2}$

$$\underbrace{\frac{-\frac{1}{2}}{x+3}}_{\downarrow\, 0} \le \underbrace{\frac{\frac{1}{2}\sin 2x}{x+3}}_{\Downarrow\, T.C.\ 0} \le \underbrace{\frac{\frac{1}{2}}{x+3}}_{\downarrow\, 0}$$

- $\lim_{x\to+\infty} \frac{x+\sin x}{4x-\cos x} = \frac{1}{4}$

$$\underbrace{\frac{x-1}{4x+1}}_{\downarrow\, 1/4} \le \underbrace{\frac{x+\sin x}{4x-\cos x}}_{\Downarrow\, T.C.\ 1/4} \le \underbrace{\frac{x+1}{4x-1}}_{\downarrow\, 1/4}$$

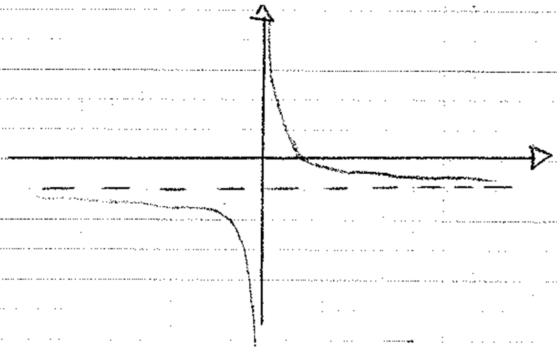
- $\lim_{x\to 0} [1-2x^2] = 0$

- $\lim_{x\to+\infty} \frac{[1-2x^2]}{x^2+3} = -2$

$x-1 \le [x] \le x$

$$\underbrace{\frac{-2x^2}{x^2+3}}_{\downarrow\, -2} \le \underbrace{\frac{[1-2x^2]}{x^2+3}}_{\Downarrow\, T.C.\ -2} \le \underbrace{\frac{1-2x^2}{x^2+3}}_{\downarrow\, -2}$$

- $\lim_{x\to+\infty} M\left(\frac{1-x}{x}\right)=0$



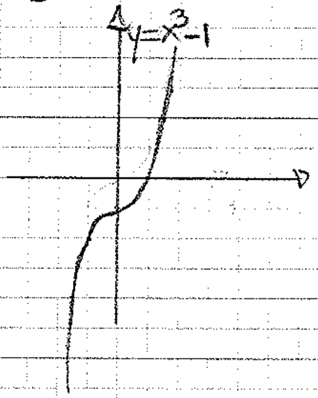
$$-1<\frac{1-x}{x}<0 \quad \forall x>1$$
$$0<M\left(\frac{1-x}{x}\right)<1 \quad \forall x>1$$
$$0<\frac{1}{x}<1$$

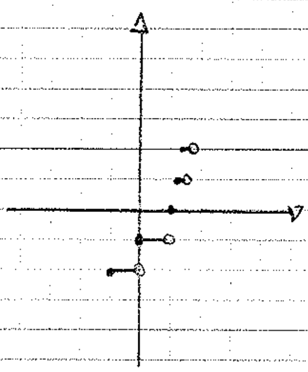
$$M\left(\frac{1-x}{x}\right)=\frac{1-x}{x}-\left[\frac{1-x}{x}\right]=\frac{1-x}{x}-(-1)=\frac{1}{x}$$

- $\lim_{x\to-\infty} \frac{1}{\underbrace{x^{1/3}}_{\sqrt[3]{x}}(3-\cos x)}=0$

$$-1\le \cos x\le 1$$
$$2\le 3-\cos x\le 4$$
$$2\sqrt[3]{x}\le \sqrt[3]{x}(3-\cos x)\le 4\sqrt[3]{x}$$
$$\downarrow \qquad \Downarrow \qquad \downarrow$$
$$-\infty \qquad -\infty \qquad -\infty$$

- $\lim_{x\to 1}\left[x^3-1\right]$ non esiste





CONFRONTO LOCALE (LANDAU)

$x_0: \{c\in\mathbb{R}, c^+, c^-, \pm\infty\}$

Se $f(x)$ e $g(x)$ sono definite su $\mathring{I}_{x_0}$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\ell \rightarrow \ell=0 \Rightarrow \underset{x\to x_0}{f(x)=o(g(x))} \quad \text{con } g(x)\neq 0 \ \forall x\in \mathring{I}_{x_0}$$
$$\rightarrow \ell=1 \Rightarrow \underset{x\to x_0}{f(x)\sim g(x)}$$

<u>PROPRIETÀ</u>:

- $x\to x_0 \quad f\sim g \Rightarrow f=g+o(g)$
- $x\to x_0 \quad f(x)=o(1) \Rightarrow \lim_{x\to x_0} f(x)=0 \quad f(x)$ infinitesima

- $x \to x_0 \quad o(f(x)) = o(k f(x))$
- $x \to 0 \quad x^m = o(x^n) \iff m > n$

  $x \to +\infty \quad x^m = o(x^n) \iff m < n$

- se su $\mathring{I}_{x_0}$ si ha $\begin{matrix} f \sim f_1 \\ g \sim g_1 \end{matrix} \Longrightarrow \lim_{x \to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f_1}{g_1}$

  $\lim_{x \to x_0} f \cdot g = \lim_{x \to x_0} f_1 \cdot g_1$

- $\lim_{x \to x_0} \frac{f(x) + o(f(x))}{g(x) + o(g(x))} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$
- $x \to 0 \quad o(x^2) + o(x) = o(x)$

  $x \to +\infty \quad o(x^2) + o(x) = o(x^2$

LIMITI FONDAMENTALI:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \iff \sin x \underset{x \to 0}{\sim} x \iff \sin x \underset{x \to 0}{=} x + o(x)$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \quad \lim_{x \to 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = 1 \iff (1 - \cos x) \underset{x \to 0}{\sim} \left(\frac{1}{2}x^2\right)$$

$$\iff 1 - \cos x = \frac{1}{2}x^2 + o\left(\frac{1}{2}x^2\right)$$

....

$$\boxed{\begin{array}{ll}
\sin x = x + o(x) & x \to 0 \\
\cos x = 1 - \frac{1}{2}x^2 + o(x^2) & x \to 0 \\
e^x = 1 + x + o(x) & x \to 0 \\
\ln(1+x) = x + o(x) & x \to 0 \\
(1+x)^\alpha = 1 + \alpha x + o(x) & x \to 0 \\
\ln x = o(x^k) \quad k > 0 & x \to +\infty \\
\operatorname{arctg} x = x + o(x) & x \to 0 \\
\arcsin x = x + o(x) & x \to 0 \\
\operatorname{tg} x = x + o(x) & x \to 0
\end{array}}$$

- per $x \to x_0$ se $\varphi \to 0 \Longrightarrow e^\varphi = 1 + \varphi + o(\varphi)$

  con $\varphi \to 0$ x oppure φ?

  È VALIDO ANCHE CON GLI ALTRI LIMITI FONDAMENTALI!

φ : PARTE PRINCIPALE

se $\varphi = k x^m$ $\quad m$: ORDINE DI INFINITO $\quad x \to +\infty$

INFINITESIMO $\quad x \to 0$

esempi	$x \to 0$	$x \to +\infty$
$5+x+x^4$	~ 5	$\sim x^4$
$x-\frac{1}{2}x^2$	$\sim x$	$\sim -\frac{1}{2}x^2$
$x+x^4$	$\sim x$	$\sim x^4$
x^4	$\sim x^4$	$\sim x^4$

$\boxed{x}$

- $e^{x^2} \quad x \to 0$

  $\varphi = x^2 \to 0$

  $e^{x^2} = 1 + x^2 + o(x^2) \quad x \to 0$

- $\ln(1+3x^3) \quad x \to 0$

  $\varphi = 3x^3 \to 0$

  $\ln(1+3x^3) = 3x^3 + o(x^3) \quad x \to 0$

- $e^{x-1} \quad x \to 0$

  $\varphi = x-1 \to -1$ NO!

- $e^{x-1} \quad x \to 1$

  $\varphi = x-1 \to 0$

  $e^{x-1} = 1 + (x-1) + o(x-1) \quad x \to 1$

- $\ln\left(1+\frac{x^2}{2}\right) = \frac{x^2}{2} + o(x^2) \quad x \to 0$

  $\ln\left(1+\frac{x^2}{2}\right) \sim \frac{x^2}{2} \quad x \to 0$

  $f(x) = \ln\left(1+\frac{x^2}{2}\right) \quad x \to 0$

  parte principale $= \frac{x^2}{2}$

  infinitesimo di ordine 2

---

x

$$\lim_{x\to 0} \frac{\sqrt{\cos x} - 1}{\sin 2x\left(\sqrt{1+3x}-1\right)} = \lim_{x \to 0} \frac{-\frac{1}{4}x^2}{3x^2} = -\frac{1}{12}$$

- $N(x) = \sqrt{\cos x} - 1 = \sqrt{1-\frac{1}{2}x^2+o(x^2)} - 1 =$ $\qquad \varphi = -\frac{1}{2}x^2 + o(x^2) \to 0$

  $= \sqrt{1+\varphi} - 1 = \cancel{1} + \frac{1}{2}\varphi + o(\varphi) - \cancel{1} =$

  $= -\frac{1}{4}x^2 + o(x^2) + o\left(-\frac{1}{2}x^2 + \cancel{o(x^2)}\right) =$

  $= -\frac{1}{4}x^2 + o(x^2)$

  $\sqrt{\cos x} - 1 \sim -\frac{1}{4}x^2 \quad x \to 0$

- $D_1(x) = \sin 2x = 2x + o(x) \qquad \beta = 2x \to 0$

- $D_2(x) = \sqrt{1+3x} - 1 = 1 + \frac{1}{2}\cdot 3x + o(x) = 1 + \frac{3}{2}x + o(x)$

  $\theta = 3x \to 0$

  $\sin 2x \left(\sqrt{1+3x} - 1\right) \sim 2x\left(1+\frac{3}{2}x - 1\right) \sim 3x^2$

A QUESTO PUNTO SOSTITUISCO LE FORME EQUIVALENTI NEL LIMITE

$$\lim_{x\to 0} \frac{e^{\sqrt[3]{1+2x^2}} - e}{(e^{e^x} - e^{\cos x}) \cdot \operatorname{tg} x} = \lim_{x\to 0} \frac{\frac{2}{3}\cancel{e}x^2}{\cancel{e}x^2} = \frac{2}{3}$$

- $D_2(x) = \operatorname{tg} x = x + o(x)$

  $\operatorname{tg} x \sim x$

- $D_1(x) = e^{e^x} - e^{\cos x} = e^{1+x+o(x)} - e^{1-\frac{1}{2}x^2+o(x^2)} =$

  $= e^1 \left\{ e^{x+o(x)} - e^{-\frac{1}{2}x^2+o(x^2)} \right\} =$

  $= e \left\{ \cancel{1} + x + o(x) + \cancel{o(x+o(x))} - \cancel{1} + \frac{1}{2}x^2 + o(x^2) + \cancel{o(\ldots)} \right\} =$

  $= e\,(x + o(x)) = ex + e\,o(x)$

  $(e^{e^x} - e^{\cos x}) \sim ex$

- $N(x) = e^{\sqrt[3]{1+2x^2}} - e = e^{1+\frac{1}{3}\cdot 2x^2 + o(x^2)} - e =$

  $= e\left[e^{\frac{2}{3}x^2+o(x^2)} - 1\right] = e\left[\cancel{1} + \frac{2}{3}x^2 + o(x^2) + \cancel{o(\ldots)} - \cancel{1}\right] =$

  $= \frac{2}{3}ex^2 + o(x^2)$

  $\sqrt[3]{1+2x^2} = (1+2x^2)^{\frac{1}{3}} = 1 + \frac{1}{3}(2x^2) + o(x^2) = 1 + \frac{2}{3}x^2 + o(x^2)$

  $(e^{\sqrt[3]{1+2x^2}} - e) \sim \frac{2}{3}ex^2$

Valutare l'ordine di infinitesimo della funzione

$$f(x) = \left[\sqrt[3]{x+1} - \sqrt[3]{x-1}\right] \cdot \ln\left(3 + \operatorname{arctg}\frac{2}{x}\right) \quad \text{per } x \to +\infty$$

$t = \frac{1}{x}$: se $x \to +\infty$ $t \to 0^+$ — infinitesimo campione: $\frac{1}{x^\alpha}$

- $\left[\sqrt[3]{\frac{1}{t}+1} - \sqrt[3]{\frac{1}{t}-1}\right] = t^{-\frac{1}{3}} \cdot \sqrt[3]{1+t} - t^{-\frac{1}{3}} \sqrt[3]{1-t} =$

  $= \frac{1}{\sqrt[3]{t}}\left(\sqrt[3]{1+t} - \sqrt[3]{1-t}\right) = \frac{1}{\sqrt[3]{t}}\left(\cancel{1} + \frac{1}{3}t + o(t) - \cancel{1} + \frac{1}{3}t + o(t)\right) =$

  $= \frac{1}{\sqrt[3]{t}}\left(\frac{2}{3}t + o(t)\right) = \frac{2}{3}t^{2/3} + o(t^{2/3}) \sim \frac{2}{3} \cdot \frac{1}{x^{2/3}}$

- $\ln(3 + \operatorname{arctg} 2t) = \ln(3 + 2t + o(t)) = \ln\left[3\left(1 + \frac{2}{3}t + o(t)\right)\right] =$

  $= \ln 3 + \ln\left(1 + \frac{2}{3}t + o(t)\right) = \ln 3 + \frac{2}{3}t + o(t) + \cancel{o(\ldots)} =$

  $= \ln 3 + \frac{2}{3x} + o\left(\frac{1}{x}\right) \sim \ln 3$

$f(x) \sim \frac{2}{3}\ln 3 \cdot \frac{1}{x^{2/3}}$ — ordine $= \frac{2}{3}$

LIMITI RAZIONALI IRRAZIONALI TRASCENDENTALI

- $$\lim_{x\to\pm\infty}\frac{\sqrt{2x^2+x+1}}{x-1}=\lim_{x\to\pm\infty}\frac{|x|\sqrt{2+\frac{1}{x}+\frac{1}{x^2}}}{x\left(1-\frac{1}{x}\right)}\begin{cases}x\to+\infty & \sqrt{2}\\ x\to-\infty & -\sqrt{2}\end{cases}$$

- $$\lim_{x\to0}\frac{(1-\cos x)\cdot\cot^2 x}{x^k}=\lim_{x\to0}\frac{\frac{1}{2}x^2\cdot\frac{1}{x^2}}{x^k}=\frac{1}{2x^k}=\begin{cases}\infty & k>0\\ 0 & k<0\\ 1/2 & k=0\end{cases}$$
  $k\in\mathbb{R}$

- $$\lim_{x\to+\infty}\frac{\ln(1+e^{kx})}{\sqrt{1+x^2}}=\frac{kx+o(1)}{x+o(1)}=k$$
  $k>0$

$$N(x)=\ln(1+e^{kx})=\ln\left[e^{kx}(e^{-kx}+1)\right]=kx+\ln(e^{-kx}+1)=$$
$$=kx+e^{-kx}+o(e^{-kx})=kx+o(1)$$

$$D(x)=x\sqrt{1+\frac{1}{x^2}}=x\left(1+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)=x+\frac{1}{2x}+o\left(\frac{1}{x}\right)=x+o(1)$$

- $$\lim_{x\to0^+}\frac{\ln(1+\sqrt{x})}{\sqrt[3]{x}}=\lim_{x\to0^+}\frac{\sqrt{x}+o(\sqrt{x})}{\sqrt[3]{x}}=\lim_{x\to0^+}\frac{x^{1/2}}{x^{1/3}}=$$
$$=\lim_{x\to0^+}x^{1/6}=0$$

- $$\lim_{x\to0}\frac{\ln(1+\sin^3 2x)}{2x^3}=\lim_{x\to0}\frac{\sin^3 2x+o(\sin^3 2x)}{2x^3}=\lim_{x\to0}\frac{8x^3+o(x^3)}{2x^3}=4$$

- $$\lim_{x\to+\infty}x\cdot\ln\left(2^{1/x}+\frac{1}{x\cdot\ln x}\right)=\lim_{x\to+\infty}x\cdot\left(\frac{1}{x}\ln 2+o(1)\right)=\lim_{x\to+\infty}\ln 2+o(x)$$

$$N(x)=\ln\left[2^{1/x}\left(1+\frac{1}{x\cdot2^{1/x}\ln x}\right)\right]=\ln 2^{1/x}+\underbrace{\ln\left(1+\frac{1}{x\cdot2^{1/x}\ln x}\right)}_{\to0}=\frac{1}{x}\ln 2+o(1)$$
$$\to=\frac{1}{x}\ln 2+\frac{1}{x2^{1/x}\ln x}+o\left(\frac{1}{x2^{1/x}\ln x}\right)$$
$$=\lim_{x\to+\infty}\ln 2+\underbrace{\frac{1}{2^{1/x}\ln x}}_{\to0}+\underbrace{o\left(\frac{1}{2^{1/x}\ln x}\right)}_{\to0}=\ln 2$$

- $$\lim_{x\to\pm\infty}\left(\sqrt{2x^2+x+1}-x+1\right)=\lim_{x\to\pm\infty}\left(|x|\sqrt{2+\frac{1}{x}+\frac{1}{x^2}}-x+1\right)\begin{cases}\lim_{x\to-\infty}\left(-x\sqrt{\dots}-x+1\right)=+\infty\\ \lim_{x\to+\infty}\left[x(\sqrt{2}+o(1))-x+1\right]=\end{cases}$$
$$=\lim_{x\to+\infty}\left(\sqrt{2}-1+o(x)\right)x+1=+\infty$$

- $\lim_{x\to+\infty}\left(\sqrt{x^2-1}-\sqrt{x^3+x}\right)=\lim_{x\to+\infty}\left(x\sqrt{1-\frac{1}{x^2}}-x^{3/2}\sqrt{1+\frac{1}{x^2}}\right)=\lim_{x\to+\infty}\left(x\left(1-\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)-\right.$

$\left.x^{3/2}\left(1+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)\right)=\lim_{x\to+\infty}\left(x-\frac{1}{2x}+o\left(\frac{1}{x}\right)-x^{3/2}-\frac{1}{2x^{1/2}}+o\left(\frac{1}{x^{1/2}}\right)\right)=$

$=\lim_{x\to+\infty}\left(x-x^{3/2}+o(1)\right)=\lim_{x\to+\infty}x^{3/2}\left(x^{-1/2}-1+o(1)\right)=-\infty$

- $\lim_{x\to+\infty}\left(\left(\sqrt{1+3^x}-3^x\right)\sqrt{2+3^x}\right)=\lim_{x\to+\infty}\left(3^x\left(\sqrt{\frac{1}{3^{2x}}+1}-1\right)\right)\cdot\sqrt{2+3^{2x}}=$

$=\lim_{x\to+\infty}3^x\left(\cancel{1}+\frac{1}{2\,3^{2x}}-\cancel{1}+o\left(\frac{1}{3^{2x}}\right)\right)\cdot\sqrt{2+3^{2x}}=\left(\frac{1}{2\cdot 3^x}+o\left(\frac{1}{3^x}\right)\right)\cdot 3^x\sqrt{1+\frac{2}{3^{2x}}}=$

$=\lim_{x\to+\infty}\left(\frac{1}{2}+o(1)\right)=\frac{1}{2}$

- $\lim_{x\to+\infty}\left(\sqrt{ax^2+x}-ax-b\right)=-\frac{3}{4}$ $\quad a?\ b?$

$=\lim_{x\to+\infty}\left(2x\sqrt{1+\frac{1}{4x}}-ax-b\right)=\lim_{x\to+\infty}\left(2x-\frac{1}{4}+o(1)-ax-b\right)=$

$2x\left(1-\frac{1}{8x}+o\left(\frac{1}{x}\right)\right)$

$=\lim_{x\to+\infty}\left[x(2-a)-\left(b-\frac{1}{4}\right)\right]=-\frac{3}{4}$ $\quad\begin{cases}2-a=0\\ +b-\frac{1}{4}=\frac{3}{4}\end{cases}\quad\begin{cases}a=2\\ b=1\end{cases}$

LIMITI $f(x)^{g(x)}$

$$\boxed{f(x)^{g(x)}=e^{g(x)\ln f(x)}}$$

- $\lim_{x\to0}(2-\cos x)^{\frac{1}{\sin^2 x}}=\lim_{x\to0}e^{\frac{\ln(2-\cos x)}{\sin^2 x}}=\lim_{x\to0}e^{\frac{\frac{1}{2}x^2+o(x^2)}{x^2+o(x^2)}}=e^{1/2}$

$\ln(2-\cos x)=\ln(\underbrace{2-1}_{1}+\frac{1}{2}x^2+o(x^2))=\frac{1}{2}x^2+o(x^2)$

- $\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right)^{x^2}=\lim_{t\to0^+}(\cos t)^{\frac{1}{t^2}}=\lim_{t\to0^+}e^{\frac{\ln(\cos t)}{t^2}}=\lim_{t\to0^+}e^{\frac{-\frac{1}{2}t^2+o(t^2)}{t^2}}=e^{-\frac{1}{2}}$

- $\lim_{x\to+\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x=\lim_{t\to0^+}e^{\frac{\ln(\sin t+\cos t)}{t}}=\lim_{t\to0^+}e^{\frac{\ln(t+o(t)+1-\frac{1}{2}t^2+o(t^2))}{t}}=$

$=\lim_{t\to0^+}e^{\frac{\ln(1+t+o(t))}{t}}=\lim_{t\to0^+}e^{\frac{t+o(t)}{t}}=\lim_{t\to0^+}e^{1+o(1)}=e^1$

$$\cdot \lim_{x\to 0}\left(\frac{1+\operatorname{tg}x}{1+\sin x}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}[\ln(1+\operatorname{tg}x)-\ln(1+\sin x)]} = \lim_{x\to 0} e^{\frac{1}{x}(\cancel{x}+o(x)-\cancel{x}+o(x))} = \;?$$

$$\operatorname{tg}x = x+o(x) \qquad = \lim_{x\to 0} e^{o(1)} = e^0 = 1$$

$$\sin x = x+o(x)$$

$$\cdot \lim_{x\to\frac{\pi}{2}} (\sin^2 x)^{\operatorname{tg}^2 x} = \lim_{x\to\frac{\pi}{2}} e^{\operatorname{tg}^2 x \ln(\sin^2 x)} = \lim_{t\to 0} e^{\operatorname{tg}^2(t+\frac{\pi}{2})\cdot\ln(\sin^2(t+\frac{\pi}{2}))} =$$

$$x-\frac{\pi}{2}=t$$

$$= \lim_{t\to 0} e^{+\operatorname{ctg}^2(t)\cdot\ln(\cos^2 t)} = \lim_{t\to 0} e^{+2\frac{\cos^2 t}{\sin^2 t}\ln(\cos t)} =$$

$$= \lim_{t\to 0} e^{+2\frac{1+o(1)}{t^2+o(t^2)}\cdot\left(-\frac{1}{2}t^2+o(t^2)\right)} = \lim_{t\to 0} e^{-1+o(1)} = e^{-1}$$

ORDINI INFINITESIMO / INFINITO

$$\cdot\; g(x) = \sin\frac{1}{x^2}\cdot\left[\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right] = \left[t^2+o(t^2)\right]\left[2+o(1)\right] = \left(\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)\cdot(2+o(1)) =$$

$$= \frac{2}{x^2}+o\left(\frac{1}{x^2}\right)$$

Calcolare l'ordine di infinitesimo per $x\to+\infty$

$$x=\frac{1}{t} \qquad t\to 0^+$$

$$N_1(x) = \sin\frac{1}{x^2} = \sin t^2 = t^2+o(t^2)$$

$$N_2(x) = \sqrt[3]{\frac{1}{t^3}+\frac{3}{t^2}}-\sqrt{\frac{1}{t^2}-\frac{2}{t}} = \frac{1}{t}\sqrt[3]{1+3t}-\frac{1}{t}\sqrt{1-2t} = \frac{1}{t}\left(\cancel{1}+\frac{3}{3}t+o(t)-\cancel{1}+\frac{2}{2}t+o(t)\right) =$$

$$= 2+o(1)$$

$g(x)\sim\frac{2}{x^2}$ parte principale

ordine di infinitesimo: 2

$$\cdot\; f(x) = \left(e^{\sqrt[3]{1+2x^2}}-e\right)\sin(\pi\sqrt{1+x}) \qquad x\to 0$$

$$N_1(x) = e^{\sqrt[3]{1+2x^2}}-e = e\left(e^{\frac{2}{3}x^2+o(x^2)}-1\right) = e\left(\cancel{1}+\frac{2}{3}x^2+o(x^2)-\cancel{1}\right) = \frac{2}{3}ex^2+o(x^2)\sim\frac{2}{3}ex^2$$

$$N_2(x) = \sin(\pi\sqrt{1+x}) = \sin\left(\pi\left(1+\frac{1}{2}x+o(x)\right)\right) = \sin\left(\pi+\underbrace{\frac{\pi}{2}x+o(x)}\right) = -\sin\left(\frac{\pi}{2}x+o(x)\right) =$$

$$= -\frac{\pi}{2}x+o(x)\sim-\frac{\pi}{2}x$$

$$f(x)\sim\frac{2}{3}ex^2\cdot\left(-\frac{\pi}{2}x\right) = -\frac{\pi}{3}ex^3$$

ordine infinitesimo: 3

parte principale: $-\frac{\pi}{3}ex^3$

- $f(x) = \frac{2^x - x^2}{(x-2)} \operatorname{arctg}(x-2) \cdot \ln(x-1) \qquad x \to 2$

$z = x-2 \quad z \to 0$

$$f(x) = \frac{2^{z+2} - (z+2)^2}{z} \cdot \operatorname{arctg}(z) \cdot \ln(1+z)$$

$$N_1(x) = 4 \cdot 2^z - z^2 - 4z - 4 = 4(1 + z\ln 2 + o(z)) - z^2 - 4z - 4$$

$$2^z = e^{z\ln 2} = 1 + z\ln 2 + o(z)$$

$$f(x) = \frac{\cancel{4} + 4z\ln 2 + o(z) - 4z - \cancel{4}}{z}(1 + o(1))(z + o(z)) =$$

$$= z(\ln 16 - 4)(z + o(z)) = z^2\left(\ln\frac{16}{e^4}\right) + o(z^2) \sim 4\ln\frac{2}{e} z^2$$

$4 = \ln e^4$

$pp = 4\ln\frac{2}{e}(x-2)^2$

ordine infinitesimo: 2

- $f(x) = \frac{\ln(3 + \operatorname{arctg}\frac{2}{x}) - \ln 3}{\left[\sin(\sqrt{x^2+1} - x)\right]^2} \cdot \left[e^{-\frac{x}{4x^2+1}} - 1\right]^3$

$x \to +\infty \quad t = \frac{1}{x} \quad t \to 0^+$

$$N_1(x) = \ln(3 + \operatorname{arctg} 2t) - \ln 3 = \ln\left[1 + \frac{1}{3}\operatorname{arctg} 2t\right] = \ln\left(1 + \frac{2}{3}t + o(t)\right) =$$
$$= \frac{2}{3}t + o(t) \sim \frac{2}{3}t = \frac{2}{3x}$$

$$N_2(x) = \left[e^{-\frac{\frac{1}{t}}{\frac{4}{t^2}+1}} - 1\right]^3 = \left[e^{-\frac{t}{4+t^2}} - 1\right]^3 = \left(-\frac{t}{4+t^2} + o(t)\right)^3 = \left(-\frac{t}{4} + o(t)\right)^3 \sim \left(-\frac{1}{4x}\right)^3 = -\frac{1}{4x^3}$$

$$D_1(x) = \sin^2(\sqrt{x^2+1} - x) = \left[\sin\left(\frac{1}{t}\sqrt{1+t^2} - \frac{1}{t}\right)\right]^2 = \left[\sin\left[\frac{1}{t}\left(\cancel{1} + \frac{t^2}{2} + o(t^2) - \cancel{1}\right)\right]\right]^2 =$$
$$= \left[\sin\left(\frac{t}{2} + o(t^2)\right)\right]^2 = \left[\frac{t}{2} + o(t)\right]^2 = \frac{t^2}{4} + o(t^2) \sim \frac{1}{4x^2}$$

$$f(x) \sim \frac{2}{3x} \cdot \left(-\frac{1}{4_8 x^3}\right) \cdot 4x^2 = -\frac{1}{24x^2}$$

ordine di infinitesimo: 2

pp: $-\frac{1}{24x^2}$

CONTINUITÀ

- $f(x) = \frac{x+1}{x-2} \quad x \neq 2$

$\text{dom} f = \{ \forall x \in \mathbb{R} : x \neq 2 \}$ la funzione è continua sul suo dominio

$f(x)$ è CONTINUA in x_0
$\Updownarrow$ def
- $x_0 \in \text{dom} f$
- $\lim_{x \to x_0^-} f(x) = \ell_1 \in \mathbb{R}$
  $\lim_{x \to x_0^+} f(x) = \ell_2 \in \mathbb{R}$ $\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$
  $f(x_0) = \ell_1 = \ell_2$

STUDIARE LA CONTINUITÀ

- $f(x) = \sqrt{x^2 - 4}$
  - $x^2 - 4 \geq 0 \to x \leq -2 \vee x \geq 2$
    $\text{dom} f = (-\infty, -2] \cup [2, +\infty)$
  La funzione è CONTINUA (su tutto il suo dominio)
  ↳ composizione di funzioni elementari continue

- $\lim_{x \to 0} \sqrt[6]{x^6 - x^2} = 0 \overset{\text{def}}{\Longleftrightarrow} \forall \varepsilon > 0, \exists \delta > 0 / 0 < |x - 0| < \delta$
  $\Rightarrow |\sqrt[6]{x^6 - x^2} - 0| < \varepsilon$
  - $x^2 (x^2 - 1) \geq 0 \to x = 0 \vee x \leq -1 \vee x \geq 1$
    $\text{dom} f = (-\infty, -1] \cup \underbrace{[0, 0]}_{\{0\}} \cup [1, +\infty)$
  Il limite non esiste $\to$ non c'è un intorno di $x_0 = 0$
  $x_0 = 0$ punto di discontinuità! $\to$ xkè il limite non esiste

- $f(x) = \sqrt[4]{\frac{x+1}{x}}$
  - $N > 0 \quad x \geq -1$
    $D > 0 \quad x > 0$

	-1		0	
$-$		$+$		$+$
$-$		$-$		$+$
$+$	$\bullet$	$-$	$\nexists$	$+$

$x \leq -1$
$x > 0$

$\text{dom} f = (-\infty, -1] \cup (0, +\infty)$

- $x = -1 \in \text{dom} f$
  $\lim_{x \to -1^-} f(x) = 0 \quad f(-1) = 0 \Rightarrow f(x)$ CONTINUA (a sinistra) in $x = -1$

LA FUNZIONE È CONTINUA

- $f(x) = \begin{cases} 1 & x=0 \\ \frac{1}{2-e^{1/x}} & x \neq 0 \end{cases}$

  - $\begin{cases} x \neq 0 \\ 2-e^{1/x} \neq 0 \end{cases} \begin{cases} x \neq 0 \\ x \neq \frac{1}{\ln 2} \end{cases}$

    $\text{dom} f = (-\infty, \frac{1}{\ln 2}) \cup (\frac{1}{\ln 2}, +\infty)$

  - $x=0 \in \text{dom} f$

    $\lim_{x \to 0^+} \frac{1}{2-e^{1/x}} = 0 \qquad f(0) = 1$

    $\lim_{x \to 0^-} \frac{1}{2-e^{1/x}} = \frac{1}{2}$

    $f(x)$ non è continua in $x=0 \to f(x)$ è continua $\forall x \in \text{dom} f / x \neq 0$

- $f(x) = \begin{cases} \frac{\ln x^2 - 1}{\ln x^2 + 1} & x \neq 0 \\ 2 & x=0 \end{cases}$

  - $\begin{cases} x^2 \neq 0 \\ \ln x^2 \neq -1 \end{cases} \begin{cases} x \neq 0 \\ x^2 \neq e^{-1} \end{cases} \begin{cases} x \neq 0 \\ x \neq \pm\sqrt{e^{-1}} \end{cases}$

    $\text{dom} f = (-\infty, -e^{-\frac{1}{2}}) \cup (-e^{-\frac{1}{2}}, e^{-\frac{1}{2}}) \cup (e^{-\frac{1}{2}}, +\infty)$

  - $x=0 \in \text{dom} f$

    $\lim_{x \to 0^-} \frac{\ln x^2 (1 - \frac{1}{\ln x^2})}{\ln x^2 (1 + \frac{1}{\ln x^2})} = 1$

    $\lim_{x \to 0^+} f(x) = 1$ $\quad$ > $f(x)$ non è continuo in $x=0$

    $f(0) = 2$

    $f(x)$ è CONTINUA $\forall x \in \text{dom} f / x \neq 0$

- $f(x) = \begin{cases} \frac{\sqrt{2-\sqrt{x+1}}}{1-x} + 2^{-\frac{1}{x^2}} & x \neq 0, x \neq 1 \\ 1 & x=0 \lor x=1 \end{cases}$

  - $\begin{cases} x+1 \geq 0 \\ 2-\sqrt{x+1} \geq 0 \end{cases} \begin{cases} x \geq -1 \\ \sqrt{x+1} \leq 2 \end{cases} \begin{cases} x \geq -1 \\ x \leq 3 \end{cases} \quad -1 \leq x \leq 3$

    $\text{dom} f = [-1, 3]$

  - $\lim_{x \to 0^-} f(x) = 1 \quad \lim_{x \to 0^+} f(x) = 1 \quad f(0) = 1$ continuo!

    $\lim_{x \to 1^-} f(x) = +\infty \quad \lim_{x \to 1^+} f(x) = -\infty \quad f(1) = 1$ non continua

    $f(x)$ CONTINUA $\forall x \in \text{dom} f / x \neq 1$

$\cdot\ f(x) = \begin{cases} \dfrac{x+k}{x+2} & x \leq 0 \\ \sqrt{x+k} & x > 0 \end{cases}$

$- \begin{cases} x+2 \neq 0 \\ x+k \geq 0 \end{cases} \begin{cases} x \neq -2 \\ x \geq -k \end{cases}$ $\quad x \geq -k$ ma $x \neq -2$ $\quad k>2$

$x > -k \quad k=2$

$x \geq -k \quad k<2$

• se $-k<0 \to k>0$ $x \neq -2$

dom $f = \{ \forall x \in \mathbb{R} : x \neq -2 \}$

$\lim_{x \to 0^-} \dfrac{x+k}{x-2} = \dfrac{k}{2}$

$\lim_{x \to 0^+} \sqrt{x+k} = \sqrt{k}$ $\quad \begin{cases} \frac{k}{2} = \sqrt{k} \\ k>0 \end{cases} \begin{cases} k^2 - 4k = 0 \\ k>0 \end{cases} \begin{cases} k(k-4)=0 \\ k>0 \end{cases} \quad k=4$

$f(0) = \dfrac{k}{2}$

Per $\boxed{k=4}$ $f(x)$ è continua sul suo dominio

• se $k=0$

$\lim_{x \to 0^-} \dfrac{x}{x+2} = 0$

$\lim_{x \to 0^+} \sqrt{x} = 0$

$f(0)=0$

Per $\boxed{k=0}$ $f(x)$ è continua sul suo dominio

• se $k<0$

$\to$ non esiste il limite destro $x \to 0^+$

$f(x)$ è continua per $\forall k<0$ (in $x=0$ continua parzialmente)

$\cdot\ g(x) = \begin{cases} x+1 & x \leq 1 \\ 3-kx^2 & x>1 \end{cases}$

- dom $g = \mathbb{R}$

- $x=1 \in$ dom g

$\lim_{x \to 1^-} (x+1) = 2$

$\lim_{x \to 1^+} (3-kx^2) = 3-k$ $\quad 3-k=2$

$k=1$

$g(1)=2$

Per $\boxed{k=1}$ $f(x)$ è continua nel suo dominio

DERIVABILITÀ

- Determinare la derivata con la definizione di
$$f(x)=\sqrt{x^2+1} \quad \text{in } x_0=1$$
$$\lim_{x\to x_0}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{2}}{x-1}=\lim_{t\to 0}\frac{\sqrt{(t+1)^2+1}-\sqrt{2}}{t}=\lim_{t\to 0}\frac{\sqrt{2}+\frac{t}{\sqrt{2}}+o(t)-\sqrt{2}}{t}=$$
$$=\lim_{t\to 0}\left(\frac{1}{\sqrt{2}}+o(1)\right)=\frac{1}{\sqrt{2}}=f'(1)$$
$$\sqrt{(t+1)^2+1}=\sqrt{t^2+2+2t}=\sqrt{2}\left[\sqrt{\frac{t^2}{2}+t+1}\right]=\sqrt{2}\left[1+\frac{1}{4}t^2+\frac{1}{2}t+o(t)\right]=\sqrt{2}\left[1+\frac{t}{2}+o(t)\right]$$

- Determinare usando la definizione la derivata di
$g(x)=\sqrt[3]{x^2+2}$ nel punto x_0
$$\frac{\Delta g}{\Delta x}=\frac{g(x)-g(x_0)}{x-x_0}=\frac{\sqrt[3]{x^2+2}-\sqrt[3]{x_0^2+2}}{x-x_0}=\frac{\sqrt[3]{(x_0+t)^2+2}-\sqrt[3]{x_0^2+2}}{t}$$
$$t=x-x_0\to 0$$
$$\sqrt[3]{(x_0+t)^2+2}=\sqrt[3]{x_0^2+t^2+2x_0t+2}=\sqrt[3]{(x_0^2+2)}\cdot\sqrt[3]{1+\frac{t^2+2x_0t}{x_0^2+2}}=$$
$$=\sqrt[3]{x_0^2+2}\left[1+\frac{1}{3}\left(\frac{t^2+2x_0t}{x_0^2+2}\right)+o(t)\right]=\sqrt[3]{x_0^2+2}\left[1+\frac{2}{3}\frac{x_0}{x_0^2+2}t+o(t)\right]$$
$$\lim_{x\to x_0}\frac{\Delta g}{\Delta x}=\lim_{t\to 0}\frac{\sqrt[3]{(x_0+t)^2+2}-\sqrt[3]{x_0^2+2}}{t}=\lim_{t\to 0}\left\{\frac{2}{3}\frac{x_0}{(x_0^2+2)^{2/3}}\cdot\frac{\cancel{t}}{\cancel{t}}+o(1)\right\}=\frac{2}{3}\frac{x_0}{(x_0^2+2)^{2/3}}=g'(x_0)$$

- $f(x)=(x^2-x^4)^{1/5}+x^{1/3}$
  - $\text{dom} f=\mathbb{R}$
  - $f(x)$ è continua sul suo dominio
  - derivabilità: in $x^2(1-x^2)=0 \to x=0,\ x=\pm 1$ a rischio di derivabilità

$$x=0 \quad \lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{(x^2-x^4)^{1/5}+x^{1/3}}{x}=\lim_{x\to 0}\left(\frac{x^{2/5}(1-x^2)^{1/5}}{x}+\frac{x^{1/3}}{x}\right)=$$
$$=\lim_{x\to 0}\left(\frac{(1-x^2)^{1/5}}{x^{3/5}}+\frac{1}{x^{2/3}}\right)=\lim_{x\to 0}\frac{1}{x^{2/3}}\left((1-x^2)^{1/5}\cdot x^{1/15}+1\right)\ \begin{cases}0^+\to+\infty\\0^-\to+\infty\end{cases}$$
flesso a tg verticale

$$x=1 \quad \lim_{x\to 1}\frac{\boxed{(x^2-x^4)^{1/5}}\boxed{+x^{1/3}-1}}{x-1}=\lim_{t\to 0}\left(\frac{-(t+1)^{2/5}\cdot t^{1/5}\cdot(2+t)^{1/5}}{t}+\frac{(t+1)^{1/3}-1}{t}\right)=$$
$$x-1=t\to 0$$
$$=\lim_{t\to 0}\left(-\frac{(t+1)^{2/5}(t+2)^{1/5}}{t^{4/5}}+\frac{\cancel{1}+\frac{1}{3}t+o(t)-\cancel{1}}{t}\right)=\lim_{t\to 0}\left(-\frac{(t+1)^{2/5}(t+2)^{1/5}}{t^{4/5}}+\frac{1}{3}+o(1)\right)=$$
$$=-\frac{\sqrt[5]{2}}{0^+}+\frac{1}{3}=-\infty$$
punto di non derivabilità

- $f(x) = (x-1)^2 |x-2|^{1/3}$

  - $\mathrm{dom} f = \mathbb{R}$
  - $f(x)$ è continuo su $\mathbb{R}$
  - $x = 2$

$$\lim_{x\to 2} \frac{\Delta f}{\Delta x} = \lim_{x\to 2} \frac{(x-1)^2|x-2|^{1/3}}{x-2} = \lim_{t\to 0} \frac{(1+t)^2|t|^{1/3}}{t} \begin{cases} \lim_{t\to 0^-} -\frac{(1+t)^2}{t^{2/3}} = -\infty & \text{cuspide} \\ \lim_{t\to 0^+} \frac{(1+t)^2}{t^{2/3}} = +\infty & \text{minimo rel.} \end{cases}$$

$x-2 = t \to 0$

- $g(x) = |x-2|^{x-2} = e^{(x-2)\ln|x-2|}$

  - $|x-2| > 0 \quad x \neq 2$

    $\mathrm{dom}\, g = (-\infty, 2) \cup (2, +\infty)$
  - $g(x)$ continua sul dom g
  - $g(x)$ derivabile sul dom g

- $\bar{g} = \begin{cases} g(x) & x \neq 2 \\ k & x = 2 \end{cases}$

  - $\mathrm{dom}\, \bar{g} = \mathbb{R}$
  - $x = 2 \in \mathrm{dom}\, \bar{g}$

$$\lim_{x\to 2^-} e^{(x-2)\ln|x-2|} = \lim_{t\to 0^-} e^{t\ln(-t)} = e^0 = 1$$

$$\lim_{x\to 2^+} e^{(x-2)\ln|x-2|} = \lim_{t\to 0^+} e^{t\ln(t)} = e^0 = 1$$

$\bar{g}(2) = k$

$g(x)$ continua su $\mathbb{R}$ per $\boxed{k=1}$

  - derivabilità: in $x = 2$

$$\lim_{x\to 2} \frac{e^{(x-2)\ln|x-2|} - 1}{x-2} = \lim_{t\to 0} \frac{e^{t\ln|t|} - 1}{t} = \lim_{t\to 0} \frac{\cancel{1} + t\ln|t| + o(t\ln|t|) - \cancel{1}}{t} =$$

$$= \lim_{t\to 0} \frac{t\ln|t|}{t} = \lim_{t\to 0} \ln|t| = -\infty$$

- $f(x) = \cos|x-3|$

  - $\mathrm{dom} f = \mathbb{R}$
  - $f(x)$ continua su $\mathbb{R}$
  - derivabilità in $x = 3$

$$\lim_{x\to 3} \frac{\cos|x-3| - 1}{x-3} = \lim_{t\to 0} \frac{\cos|t| - 1}{t} \begin{cases} \lim_{t\to 0^-} \frac{\cos(-t)-1}{t} = \lim_{t\to 0^-} \frac{1 - \frac{1}{2}t^2 + o(t^2) - 1}{t} = 0 \\ \lim_{t\to 0^+} \frac{\cos t - 1}{t} = \frac{1 - \frac{1}{2}t^2 + o(t^2) - 1}{t} = 0 \end{cases}$$

$f(x)$ derivabile su $\mathbb{R}$

$f'(3) = 0$

$$\cdot\ f(x)=\begin{cases} x+\sin x & x\le \pi \\ x^3+k(x-\pi)^2+hx & x>\pi \end{cases}$$

- $\operatorname{dom} f=\mathbb{R}$
- $x=\pi\in \operatorname{dom} f$

$$\lim_{x\to\pi^-}(x+\sin x)=\pi$$
$$\lim_{x\to\pi^+}(x^3+k(x-\pi)^2+hx)=\pi^3+h\pi \quad \Rightarrow \quad \pi=\pi^3+h\pi \quad h=1-\pi^2$$
$$f(\pi)=\pi$$

- derivabilità in $x=0$

$$\lim_{x\to\pi^-}\frac{x+\sin x-\pi}{x-\pi}=\lim_{t\to0^-}\frac{t+\cancel{\pi}-\cancel{\pi}-\sin t}{t}=\lim_{t\to0^-}\left(1-\frac{\sin t}{t}\right)=0$$

$$\lim_{x\to\pi^+}\frac{x^3+k(x-\pi)^2+(1-\pi^2)x-\pi}{x-\pi}=\lim_{t\to0^+}\frac{(t+\pi)^3+kt^2+t-\pi^2t+\cancel{\pi}-\pi^3-\cancel{\pi}}{t}=$$

$$=\lim_{t\to0^+}\frac{t^3+\cancel{\pi^3}+3\pi t^2+3\pi^2t+kt^2+t-\pi^2t-\cancel{\pi^3}}{t}=\lim_{t\to0^+}\left(t^2+t(3\pi+k)+2\pi^2+1\right)=$$

$=2\pi^2+1$ non esiste il limite $\to \forall k$ $f'(\pi)$ non esiste

ASINTOTI

- asintoto verticale in $x=x_0 \xleftrightarrow{\text{def}} \lim_{x\to x_0} f(x)=\infty$
- asintoto orizzontale $y=y_0 \xleftrightarrow{\text{def}} \lim_{x\to\pm\infty} f(x)=y_0$
- asintoto obliquo $y=mx+q \xleftrightarrow{\text{def}} \begin{cases} m=\lim_{x\to\pm\infty}\frac{f(x)}{x} \overset{?}{=} \lim_{x\to\pm\infty} f'(x) \\ q=\lim_{x\to\pm\infty}[f(x)-mx]\end{cases}$

vale solo se esiste il limite

- Date 2 funzioni $f(x)$ e $g(x)$:

① $f\sim g$ $x\to+\infty \xleftrightarrow{\text{def}} \lim_{x\to+\infty}\frac{f(x)}{g(x)}=1$

② f è asintotica a g $x\to+\infty \xleftrightarrow{\text{def}} \lim_{x\to+\infty}|f(x)-g(x)|=0$

① $\not\Leftrightarrow$ ②

① $f(x)=g(x)+o(g(x))$ $x\to+\infty$ equivalenti

② $f(x)=g(x)+o(1)$ $x\to+\infty$ asintotica

- $f(x) = e^x \qquad g(x) = e^x + \frac{1}{x} \qquad h(x) = e^x + 1$

  - f, g: $\quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{e^x}{e^x+\frac{1}{x}} = 1$

    $\lim_{x\to+\infty} \left(f(x) - g(x)\right) = \lim_{x\to+\infty} \left(\cancel{e^x} - \cancel{e^x} - \frac{1}{x}\right) = \lim_{x\to+\infty} \left(-\frac{1}{x}\right) = 0$

    f e g sono equivalenti ed asintotiche per $x \to +\infty$

  - f, h: $\quad \lim_{x\to+\infty} \frac{f(x)}{h(x)} = \lim_{x\to+\infty} \frac{e^x}{e^x+1} = 1$

    $\lim_{x\to+\infty} \left[f(x) - g(x)\right] = \lim_{x\to+\infty} \left[\cancel{e^x} - \cancel{e^x} - 1\right] = -1$

    f e h sono equivalenti

- $y = |x|\, e^{\frac{x-1}{x+1}}$

  - $\mathrm{dom} f = \mathbb{R} \setminus \{-1\}$
  - $\lim_{x\to-1^-} |x|\, e^{\frac{x-1}{x+1}} = +\infty \qquad x = -1$

    $\lim_{x\to-1^+} |x|\, e^{\frac{x-1}{x+1}} = 0 \qquad$ asintoto verticale sinistro

  - $\lim_{x\to-\infty} |x|\, e^{\frac{x-1}{x+1}} = +\infty \qquad$ asintoto obliquo

    $\lim_{x\to+\infty} |x|\, e^{\frac{x-1}{x+1}} = +\infty \qquad y = ex - 2e$ per $x \to +\infty$

    $y = -ex + 2e$ per $x \to -\infty$

  - $m = \lim_{x\to+\infty} \frac{\cancel{|x|}\, e^{\frac{x-1}{x+1}}}{\cancel{x}} = e$

    $m' = \lim_{x\to-\infty} \frac{-|x|\, e^{\frac{x-1}{x+1}}}{x} = -e$

    $q = \lim_{x\to+\infty} \left[|x|\, e^{\frac{x-1}{x+1}} - ex\right] = \lim_{t\to0^+} \left[\frac{1}{t} e^{1-\frac{2t}{1+t}} - \frac{e}{t}\right] =$

    $= \lim_{t\to0^+} \left[\frac{e}{t}\left(e^{-\frac{2t}{1+t}} - 1\right)\right] = \lim_{t\to0^+} \frac{e}{t}\left(\cancel{1} - \frac{2t}{1+t} + o(\_) - \cancel{1}\right) = \lim_{t\to0^+} \left[-\frac{2e}{1+t} + o(1)\right] = -2e$

    $q' = \lim_{x\to-\infty} \left[-x \cdot e^{\frac{x-1}{x+1}} + ex\right] = \lim_{t\to0^-} \left[-\frac{1}{t} e^{1-\frac{2t}{1+t}} - \frac{e}{t}\right] = \lim_{t\to0^-} \frac{e}{t}\left[-\frac{2t}{1+t} + o(t)\right] = 2e$

- $y = x + \operatorname{arctg}|x-1|$

  - $\mathrm{dom} f = \mathbb{R}$
  - $\lim_{x\to-\infty} \left(x + \operatorname{arctg}|x-1|\right) = -\infty$

    $\lim_{x\to+\infty} \left(x + \operatorname{arctg}|x-1|\right) = +\infty$

    $m = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left[1 + \frac{\overbrace{\operatorname{arctg}|x-1|}^{\pi/2}}{x}\right] = 1$

    $q = \lim_{x\to\pm\infty} \left[x + \operatorname{arctg}|x-1| - x\right] = \lim_{x\to\pm\infty} \operatorname{arctg}|x-1| = \frac{\pi}{2}$

    asintoto obliquo completo $y = x + \frac{\pi}{2}$

- $y = x + \sqrt{x^2+1}$

- dom f = $\mathbb{R}$
- $\lim_{x\to+\infty} f(x) = +\infty$

$\lim_{x\to-\infty}\left[x - x\sqrt{1+\frac{1}{x^2}}\right] = \lim_{x\to-\infty} x\left(1-\sqrt{1+\frac{1}{x^2}}\right) = \lim_{x\to-\infty} x\left(\cancel{1}-\cancel{1}-\frac{1}{2x}+o\left(\frac{1}{x^2}\right)\right) =$

$= \lim_{x\to-\infty}\left(-\frac{1}{2x}+o\left(\frac{1}{x}\right)\right) = 0$

$y=0$ asintoto orizzontale sinistro

$m = \lim_{x\to+\infty}\frac{x+\sqrt{x^2+1}}{x} = \lim_{x\to+\infty}\left(1+\frac{x\sqrt{1+\frac{1}{x^2}}}{x}\right) = 2$

$q = \lim_{x\to+\infty}\left(x+\sqrt{x^2+1}-2x\right) = \lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right) =$

$= \lim_{x\to+\infty}\left(x\sqrt{1+\frac{1}{x^2}}-x\right) = \lim_{x\to+\infty}\left(x\left[1+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right]-x\right) =$

$= \lim_{x\to+\infty}\left(\cancel{x}+\frac{1}{2x}+o\left(\frac{1}{x}\right)-\cancel{x}\right) = 0$

$y=2x$ asintoto obliquo destro

- $y = \sqrt[5]{x^5+4x}$

- dom f = $\mathbb{R}$
- $\lim_{x\to+\infty} f(x) = +\infty$

$\lim_{x\to-\infty} f(x) = -\infty$

$m = \lim_{x\to+\infty}\frac{x\sqrt{1+\frac{4}{x^4}}}{x} = 1 \qquad m' = \lim_{x\to-\infty}\frac{x\sqrt{1+\frac{4}{x^4}}}{x} = 1$

$q = \lim_{x\to\pm\infty}\left(x\sqrt[5]{1+\frac{4}{x^4}}-x\right) = \lim_{x\to\pm\infty}\left[x\left(1+\frac{4}{5x^4}+o\left(\frac{1}{x^4}\right)\right)-x\right] =$

$= \lim_{x\to\pm\infty}\left[\frac{4}{5x^3}+o\left(\frac{1}{x^3}\right)\right] = 0$

$y=x$ asintoto obliquo completo

STUDIO DI FUNZIONE

- $f(x) = \sqrt[3]{(2x-3)^2} - \sqrt[3]{x^2}$

- dom f = $(-\infty, +\infty)$
- $\lim_{x\to-\infty}\left(\sqrt[3]{(2x-3)^2}-\sqrt[3]{x^2}\right) \sim \lim_{x\to-\infty}\left(2^{2/3}x^{2/3}-x^{2/3}\right) = \lim_{x\to-\infty} x^{2/3}\left(\sqrt[3]{4}-1\right) = +\infty$

$\sqrt[3]{(2x-3)^2} \sim (2x)^{2/3}$

$\lim_{x\to+\infty}\left(\sqrt[3]{(2x-3)^2}-\sqrt[3]{x^2}\right) = \lim_{x\to+\infty} x^{2/3}\left(\sqrt[3]{4}-1\right) = +\infty$

- Intersezioni con gli assi

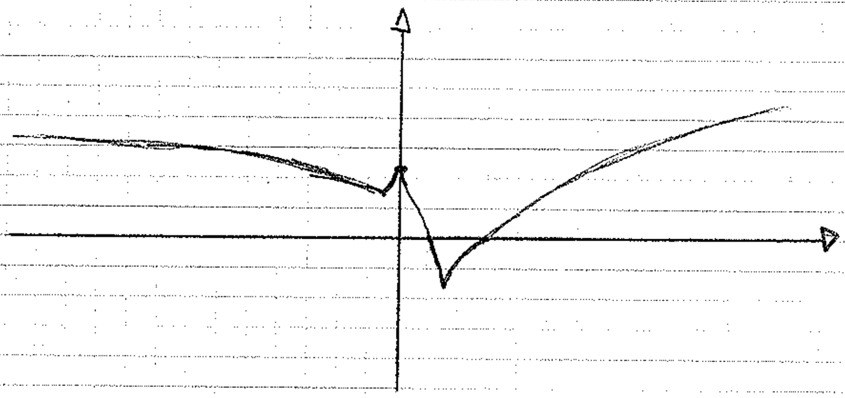
$\begin{cases} y=0 \\ y=f(x)\end{cases} \quad \sqrt[3]{(2x-3)^2} = \sqrt[3]{x^2} \quad x^2=(2x-3)^2 \quad x=\pm 2x-3 \begin{cases} x=3 \\ x=+1\end{cases}$

- asintoti obliqui

$m = \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{x^{2/3}\left(\sqrt[3]{4}-1\right)}{x} = \lim_{x\to\pm\infty}\frac{\sqrt[3]{4}-1}{x^{1/3}} = 0$ non ci sono asintoti

- continuità

f(x): somma di funzioni continue $\to$ f(x) continua su $\mathbb{R}$ (sul suo dominio)



• derivabilità: $\quad 2x-3=0 \implies x_1=\frac{3}{2} \quad f(x_1)=-\left(\frac{3}{2}\right)^{2/3}$

$\quad x=0 \implies x_2=0 \quad f(x_2)=\sqrt[3]{9}$

- $x_2=0$

$$\left(\frac{\Delta f}{\Delta x}\right)_{x_2}=\frac{f(x)-f(0)}{x-0}=\frac{\sqrt[3]{(2x-3)^2}-\sqrt[3]{2}-\sqrt[3]{9}}{x}=\frac{\sqrt[3]{9}\sqrt[3]{(\frac{2}{3}x-1)^2}-\sqrt[3]{9}}{x}-\frac{1}{x^{1/3}}$$

$$\lim_{x\to x_2}\left(\frac{\Delta f}{\Delta x}\right)_{x_2}=\lim_{x\to 0}\left[\frac{\sqrt[3]{9}\left(\sqrt[3]{(1-\frac{2}{3}x)^2}-1\right)}{x}-\frac{1}{x^{1/3}}\right]=\lim_{x\to 0}\left[\frac{\sqrt[3]{9}\left(-\frac{6}{9}x+o(x)\right)}{x}-\frac{1}{x^{1/3}}\right]=$$

$$=\lim_{x\to 0}\left[\sqrt[3]{9}\left(-\frac{6}{9}+o(1)\right)-\frac{1}{x^{1/3}}\right] \begin{cases} 0^+ \to -\infty \\ 0^- \to +\infty \end{cases}$$

cuspide verso l'alto in $(0;\sqrt[3]{9})$

- $x_1=\frac{3}{2}$

$$\left(\frac{\Delta f}{\Delta x}\right)_{x_1}=\frac{\sqrt[3]{(2x-3)^2}-\sqrt[3]{x^2}+\left(\frac{3}{2}\right)^{2/3}}{x-\frac{3}{2}}=\frac{2}{(2x-3)^{1/3}}-\frac{\sqrt[3]{x^2}-\sqrt[3]{9/4}}{x-3/2}$$

$$\lim_{x\to 3/2}\left[\frac{2}{(2x-3)^{1/3}}-\underbrace{\frac{x^{2/3}-\left(\frac{3}{2}\right)^{2/3}}{x-\frac{3}{2}}}_{\ell_2}\right]=\lim_{x\to 3/2}\left[\frac{2}{(2x-3)^{1/3}}-\left(\frac{2}{3}\right)^{4/3}\right] \begin{cases} 3/2^+ \to +\infty \\ 3/2^- \to -\infty \end{cases}$$

cuspide verso il basso in $\left(\frac{3}{2};-\left(\frac{3}{2}\right)^{2/3}\right)$

$$\ell_2=\lim_{z\to 0}\frac{\left(z+\frac{3}{2}\right)^{2/3}-\left(\frac{3}{2}\right)^{2/3}}{z}=\lim_{z\to 0}\frac{\left(\frac{3}{2}\right)^{2/3}\left[\left(1+\frac{2}{3}z\right)^{2/3}-1\right]}{z}=$$

$$=\lim_{z\to 0}\frac{\left(\frac{3}{2}\right)^{2/3}\left[1+\frac{4}{9}z+o(z)-1\right]}{z}=\left(\frac{3}{2}\right)^{2/3}\lim_{z\to 0}\left(\frac{6}{9}+o(1)\right)=\left(\frac{3}{2}\right)^{2/3}\cdot\left(\frac{2}{3}\right)^{2}=\left(\frac{2}{3}\right)^{4/3}$$

• derivata:

$$f'(x)=\frac{2}{3}(2x-3)^{-1/3}\cdot 2-\frac{2}{3}x^{-1/3}=\frac{2}{3}\left[\frac{2}{\sqrt[3]{2x-3}}-\frac{1}{\sqrt[3]{x}}\right]=\frac{2}{3}\cdot\frac{2\sqrt[3]{x}-\sqrt[3]{2x-3}}{\sqrt[3]{x(2x-3)}}$$

- $\operatorname{dom} f'=\operatorname{dom} f \setminus \{0,\frac{3}{2}\}$

- $f'\geq 0$: $\quad D>0 \quad x>\frac{3}{2},\ x<0$

$\quad N\geq 0 \quad x\geq -\frac{1}{2}$

		$-\frac{1}{2}$		0		$\frac{3}{2}$	
D	+		+	✗	−	✗	+
N	−	●	+		+		+
	−	●	+	✗	−	✗	+
	↘		↗		↘		↗

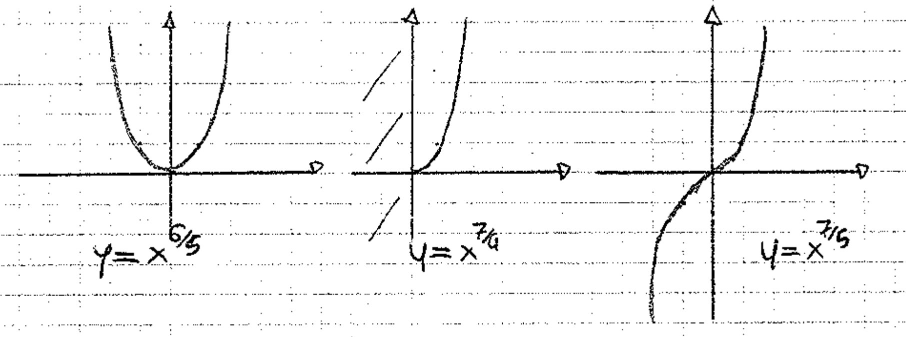
$f(x)$ strettamente crescente su $(-\frac{1}{2}, 0)$, $(\frac{3}{2}, +\infty)$
strettamente decrescente su $(-\infty, -\frac{1}{2})$, $(0, \frac{3}{2})$

$(-\frac{1}{2} ; \sqrt[3]{16} - \frac{1}{\sqrt[3]{4}})$ minimo relativo
$(\frac{3}{2} ; -(\frac{3}{2})^{2/3})$ minimo assoluto
$(0 ; \sqrt[3]{9})$ massimo relativo

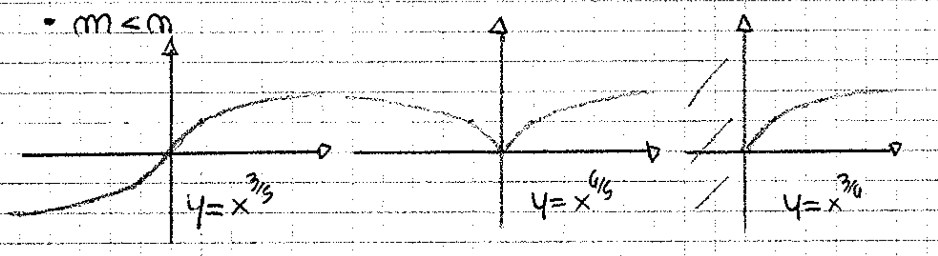
x^α

$\alpha = \frac{n}{m}$ primi tra loro

- $n > m$



- $n < m$



- $f(x) = \arctan(x-1) - |x|$

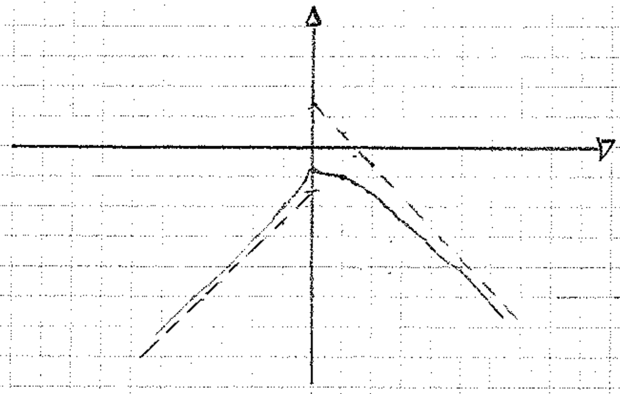
  - $\text{dom} f = (-\infty, +\infty)$
  - $\lim_{x \to \pm\infty} \left[\arctan(x-1) - |x|\right] = -\infty$ No asintoti verticali e orizzontali

$$m = \lim_{x \to \pm\infty} \left[\frac{\arctan(x-1)}{x} - \frac{|x|}{x} \right] = \mp 1 \quad \begin{matrix} +\infty \nearrow -1 \\ -\infty \searrow +1 \end{matrix}$$

$$q_{+\infty} = \lim_{x \to +\infty} \left[\arctan(x-1) - x + x\right] = \frac{\pi}{2}$$ asintoto obliquo destro $y = -x + \pi/2$

$$q_{-\infty} = \lim_{x \to -\infty} \left[\arctan(x-1) + x - x\right] = -\frac{\pi}{2}$$ asintoto obliquo sinistro $y = x - \pi/2$

- continuità: $f(x)$ continua su $\mathbb{R}$



• derivabilità: $x=0 \quad f(0)=-\frac{\pi}{4}$

$$\left(\frac{\Delta f}{\Delta x}\right)_0 = \frac{\operatorname{arctg}(x-1)-|x|+\frac{\pi}{4}}{x}$$

de l'hopital

$$\lim_{x\to 0}\left[\frac{\operatorname{arctg}(x-1)+\frac{\pi}{4}}{x}-\frac{|x|}{x}\right] \begin{cases} \xrightarrow{0^+} \dfrac{\operatorname{arctg}(x-1)+\frac{\pi}{4}}{x}-1=\frac{1}{2}-1=-\frac{1}{2} \\ \xrightarrow{0^-} \dfrac{\operatorname{arctg}(x-1)+\frac{\pi}{4}}{x}+1=\frac{1}{2}+1=\frac{3}{2} \end{cases}$$

$\left(0,-\frac{\pi+4}{4}\right)$ punto angoloso
$\hookrightarrow$ massimo

• derivata:

$$f'(x)=\frac{1}{1+(x-1)^2}-\frac{|x|}{x}=\begin{cases} \overbrace{\dfrac{1}{1+(x-1)^2}+1}^{+} & x<0 \\ \underbrace{\dfrac{1}{1+(x-1)^2}-1}_{-} & x>0 \end{cases}$$

$f(x)$ strettamente crescente su $(-\infty,0)$
strettamente decrescente su $(0,+\infty)$

$f'(x)=0 \rightarrow x=1$ flesso a tangente orizzontale in $(1,-1)$
massimo assoluto in $\left(0,-\frac{\pi}{4}\right)$

• $f(x)=2x+3\sqrt[3]{(e^x-2)^2}$

- dom $f=\mathbb{R}$
- $\lim_{x\to-\infty} f(x)=-\infty \qquad \lim_{x\to+\infty} f(x)=+\infty$

$$m=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\left(2+\frac{3\sqrt[3]{(e^x-2)^2}}{x}\right)=2$$

$y=2x+3\sqrt[3]{4}$ asintoto obliquo sinistro

$$q=\lim_{x\to-\infty}(f(x)-2x)=\lim_{x\to-\infty}\left(3\sqrt[3]{(e^x-2)^2}\right)=3\sqrt[3]{4}$$

- continuità: $f(x)$ è continua su $\mathbb{R}$

derivabilità: $e^x-2=0 \quad x_1=\ln 2$

$$\left(\frac{\Delta f}{\Delta x}\right)_{x_1}=\frac{2x+3\sqrt[3]{(e^x-2)^2}-2\ln 2}{x-\ln 2}=2+\frac{3\sqrt[3]{(e^x-2)^2}}{x-\ln 2}$$

$$\lim_{x\to\ln 2}\left[2+\frac{3\sqrt[3]{(e^x-2)^2}}{x-\ln 2}\right]=\lim_{t\to 0}\left[2+3\frac{(e^{t+\ln 2}-2)^{2/3}}{t}\right]=\lim_{t\to 0}\left[2+\frac{3}{t}\left(2(1+t+o(t))-2\right)^{2/3}\right]$$

$t=x-\ln 2\to 0$

$$= \lim_{t\to 0}\left[2+\frac{3}{t}(2t+o(t))^{2/3}\right] = 2+\lim_{t\to 0}\frac{3}{t}\cdot 2^{2/3}\cdot t^{2/3}(1+o(1))^{4/3} =$$

$$= 2+\lim_{t\to 0}\frac{3\cdot 2^{2/3}}{t^{1/3}}\cdot 1 = \begin{cases} 0^+ & +\infty \\ 0^- & -\infty \end{cases}$$

$(\ln 2, 2\ln 2)$ cuspide verso il basso → minimo relativo

- $f'(x) = 2+\cancel{3}\cdot\frac{2}{\cancel{3}}(e^x-2)^{-1/3}\cdot e^x = 2\left(1+\frac{e^x}{\sqrt[3]{e^x-2}}\right)$

$f'(x)\geq 0 \quad \frac{\sqrt[3]{e^x-2}+e^x}{\sqrt[3]{e^x-2}}\geq 0$

$N\geq 0 \quad x\geq 0$

$D>0 \quad x>\ln 2$

	0		$\ln 2$	
$-$		$+$		$+$
$-$		$-$		$+$
$+$		$-$		$+$
$\nearrow$		$\searrow$		$\nearrow$

$\sqrt[3]{e^x-2}\geq -e^x \quad z=e^x \qquad \sqrt[3]{z-2}\geq -z$

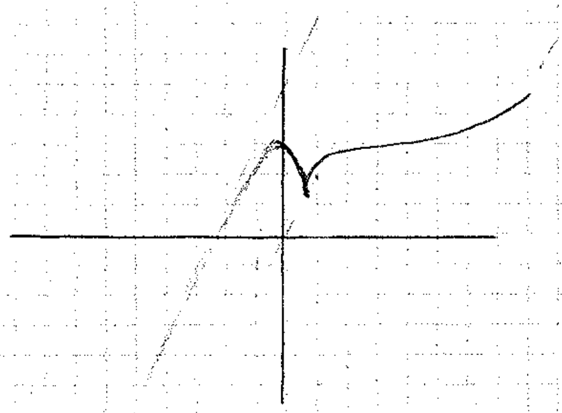
$z-2\geq -z^3$

$z^3+z-2\geq 0 \quad \underbrace{(z^2+z+2)}_{+}(z-1)\geq 0$

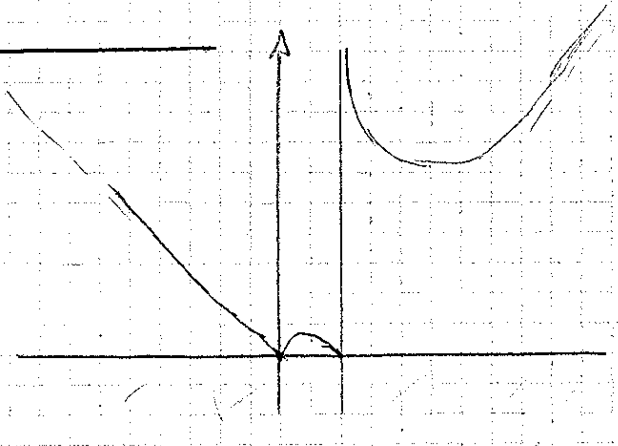
	1	0	1	-2
1		1	1	2
	1	1	2	0

$e^x-1\geq 0 \quad e^x\geq 1$
$x\geq 0$

massimo relativo in $(0,3)$



$$f(x) = |x|\cdot e^{\left(\frac{x+2}{x-2}\right)} = \begin{cases} -x\cdot e^{\frac{x+2}{x-2}} & x<0 \\ x\cdot e^{\frac{x+2}{x-2}} & x\geq 0 \end{cases}$$



- $\operatorname{dom} f = \mathbb{R}-\{2\}$

- $\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to+\infty} f(x) = +\infty$

$\lim_{x\to 2^-} f(x) = 0 \qquad \lim_{x\to 2^+} f(x) = +\infty$ $\qquad x=2$ asintoto verticale destro

$m_1 = \lim_{x\to-\infty}\frac{f(x)}{x} = -e \qquad m_2 = \lim_{x\to+\infty}\frac{f(x)}{x} = e$

$q_1 = \lim_{x\to-\infty}\left[-xe^{\frac{x+2}{x-2}}+ex\right] = -6e \quad q_2 = \lim_{x\to+\infty}\left[1x\, e^{\frac{x+2}{x-2}}-ex\right] = \lim_{x\to+\infty}\left[\frac{6ex}{x-2}+o(1)\right] = 6e$

$e^{\frac{x+2}{x-2}} = e^{\left(1+\frac{6}{x-2}\right)} = e\, e^{\frac{6}{x-2}} = e\left(1+\frac{6}{x-2}+o\left(\frac{1}{x-2}\right)\right)$

$y = ex+6e$ asintoto obliquo destro

$y = -ex-6e$ asintoto obliquo sinistro

• continuità: $f(x)$ continua su dom f

derivabilità: $x=0$

$$\left(\frac{\Delta y}{\Delta x}\right)_{x=0} = \frac{|x|}{x}\cdot e^{\frac{x+2}{x-2}} \qquad \lim_{x\to 0^+}\frac{\Delta y}{\Delta x} = \frac{1}{e} \qquad \lim_{x\to 0^-}\frac{\Delta y}{\Delta x} = -\frac{1}{e}$$

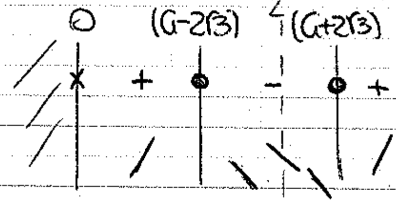
(0,0) punto angoloso → punto di minimo

$\underline{x\geq 0}$

$$f'(x) = e^{\frac{x+2}{x-2}} + x\,e^{\frac{x+2}{x-2}}\cdot\frac{x-2-x-2}{(x-2)^2} = e^{\frac{x+2}{x-2}}\left(1-\frac{4x}{(x-2)^2}\right)$$

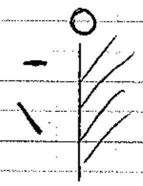
$$f'(x)\geq 0 \qquad \frac{x^2+4-4x-4x}{(x-2)^2} \qquad x^2-8x+4\geq 0$$

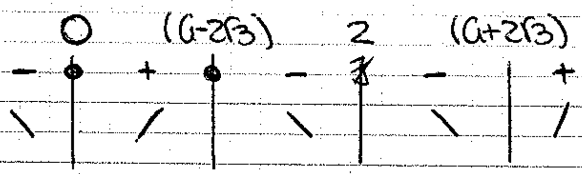
$$x = \frac{4\pm 2\sqrt{3}}{1} = 4\pm 2\sqrt{3} \quad \begin{cases} 4+2\sqrt{3} \\ 4-2\sqrt{3} \end{cases}$$



$\underline{x<0}$

$$f'(x) = e^{\frac{x+2}{x-2}}\left(\frac{4x}{(x-2)^2}-1\right)$$





minimo relativi (0,0) → assoluto

$(4+2\sqrt{3}, \ldots)$

massimo relativo $(4-2\sqrt{3}, \ldots)$

• $$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} e^{\frac{x+2}{x-2}}\,\frac{x^2-8x+4}{(x-2)^2} = 0$$

TAYLOR

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{x^4}{4!} + o(x^4)$$

$$\operatorname{tg} x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$$

$$\arcsin x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + o(x^5)$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + o(x^3)$$

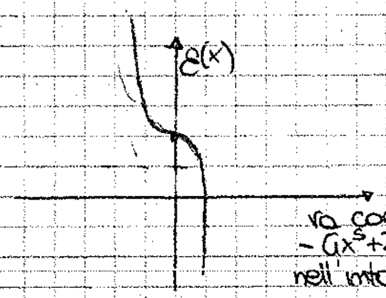
- $a_3 = \frac{f'''(0)}{3!} = \frac{1}{3} \Rightarrow \ln(1+x)$

  $f'''(0) = \frac{3!}{3} = 2$

- $g(x)$ ammette polinomio di McL. (in $x=0$)

  $P_6(x) = x^6 - 4x^5 + 2$

  - la funzione $g(x)$ passa per $(0,2)$
  - $g'(0) = g''(0) = g'''(0) = g^{(IV)}(0) = 0$
  - $\frac{g^{(V)}(0)}{5!} = -4 \to g^{(V)}(0) = -4 \cdot 5!$

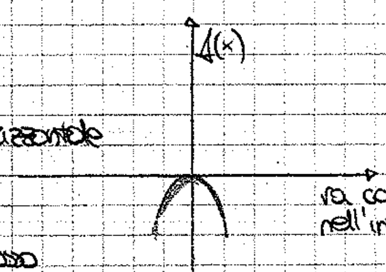


va come $-4x^5+2$ nell'intorno di 0

- $f(x)$ ha $P_5(x) = x^5 - x^4 - x^2$ (McL)

  - $f(x)$ passa per l'origine $(0,0)$
  - $f'(0) = 0$ punto critico $\to$ tangente orizzontale
  - $\frac{f''(0)}{2!} = -1$ $f''(0) = -2$

    $\hookrightarrow$ concavità verso il basso
  - $f'''(0) = 0$ $f^{(V)}(0) = 5!$



va come $-x^2$ nell'intorno di 0

- $f(x) = x^{17} - 4x^3 + x^2 - 3x - 5$ funzione polinomiale

  $\sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i = P_n(x)$

  - Trova $P_5(x)$ di McL

    $P_5(x) = -4x^3 + x^2 - 3x - 5$

    $P_2(x) = x^2 - 3x + 5$

    $\underline{P_1(x) = 3x - 5 \qquad y = 3x - 5 \text{ retta tangente in } x=0}$

- $f(x) = x^3 - 4x^2 + 1$

  McL: $P_2(x) = -4x^2 + 1$

  $P_1(x) = +1$ $y = 1$ retta tangente in $x_0 = 0$

  $P_5(x) = f(x) = P_3(x)$

TAYLOR: polinomio di Taylor di grado 2 in $x=1$

$f(1) = 1 - 4 + 1 = -2$

$f'(1) = \{3x^2 - 8x\}_1 = 3 - 8 = -5$

$f''(1) = \{6x - 8\}_1 = -2$

$P_2 = \frac{f''(1)}{2!}(x-1)^2 + \frac{f'(1)}{1!}(x-1) + \frac{f(1)}{1}(x-1)^0 = -1(x-1)^2 - 5(x-1) - 2$

$\boxed{\text{oppure}}$

$t = x - 1 \quad x = t + 1 \quad P(x) = (t+1)^3 - 4(t+1)^2 + 1 = t^3 - t^2 - 5t - 2 =$

$= (x-1)^3 - (x-1)^2 - 5(x-1) - 2$

$ML(t) \quad P_2 = -t^2 - 5t - 2 = -(x-1)^2 - 5(x-1) - 2$

- $f(x)=\begin{cases} \sin x & x\geq 0 \\ x-\frac{x^3}{6}+x^5 & x<0 \end{cases}$

→ polinomio di ML di grado massimo

Ammette $P_3(x)=x-\frac{x^3}{6}=P_4(x)$

Non ammette $P_5(x)$ → a destra e sinistra sono diversi

- $f(x)=\begin{cases} \sin x & x\leq 0 \\ x\cos x & x>0 \end{cases}$ $\quad x-\frac{x^3}{6}\ldots$ $\quad x-\frac{x^3}{2}\ldots$

Il polinomio di grado massimo di ML è di grado 2

- $y=x^{3/2}$ $\quad y=^{7/5}$ $\quad y=x^{13/5}$ $\quad y=x^{(k+\frac{m}{n})}$

- $f(x)=x^{3/2}$ $\quad f(0)=0$

$f'(x)=\frac{3}{2}x^{1/2}$ $\quad f'(0)=0$

$f''(x)=\frac{3}{4}x^{-1/2}$ $\quad$ non ammette $f''(0)$

- $f(x)=x^{13/5}$ $\quad f(0)=0$

$f'(x)=\frac{13}{5}x^{14/5}$ $\quad f'(0)=0$

$f''(x)=\frac{13}{5}\cdot\frac{16}{5}x^{9/5}$ $\quad f''(0)=0$

$f'''(x)=\frac{13}{5}\cdot\frac{16}{5}\cdot\frac{9}{5}x^{4/5}$ $\quad f'''(0)=0$

$f^{IV}(x)=\frac{13}{5}\cdot\frac{16}{5}\cdot\frac{9}{5}\cdot\frac{4}{5}\cdot x^{-1/5}$ $\quad \nexists\, f^{IV}(0)$

$y=x^{(k+\frac{m}{n})}$ con $m<n$

è derivabile fino a $f^{(k)}(x_0)$

↳ ammette polinomio k di ML

- $g(x)=\ln(1+2x^2)+\sin x^3\cdot\cos x+\operatorname{arctg} x^6+2x^{13/2}$

ammette polinomio di ML di grado 6

- $\lim_{\substack{x\to+\infty \\ t=\frac{1}{x}}} x^{\alpha}\left(e^{\operatorname{tg}(\frac{1}{x})}-e^{\frac{1}{x}}\right)=\lim_{t\to0^+}\left(\frac{1}{t}\right)^{+\alpha}\left(e^{\operatorname{tg}t}-e^{t}\right)=\lim_{t\to0^+}t^{-\alpha}\left(\cancel{1}+\cancel{t}+\cancel{\frac{1}{2}t^2}+\frac{1}{2}t^3-\cancel{t}-\cancel{\frac{t^2}{2}}-\cancel{1}-\frac{t^3}{6}+o(t^3)\right)=$

$e^{\operatorname{tg}t}=1+\frac{\operatorname{tg}t}{1!}+\frac{\operatorname{tg}^2t}{2!}+\frac{\operatorname{tg}^3t}{3!}+o(\operatorname{tg}^3t)=$

$=1+\left(t+\frac{t^3}{3}+o(t^3)\right)+\frac{1}{2}\left(t+\frac{t^3}{3}+o(t^3)\right)^2+\frac{1}{6}\left(t+o(t)\right)^3=$

$=1+t+\frac{t^3}{3}+o(t^3)+\frac{1}{2}t^2+\frac{1}{6}t^3=$

$=1+t+\frac{1}{2}t^2+\frac{1}{2}t^3+o(t^3)$

$\lim_{t\to0^+}t^{-\alpha}\frac{1}{3}t^3=\lim_{t\to0^+}\frac{1}{3}t^{3-\alpha}=\begin{cases} 0 & \alpha<3 \\ 1/3 & \alpha=0 \\ +\infty & \alpha>3 \end{cases}$

- $f(x)=(x-\sin x)\ln(1+x)$ $\quad n=6$, $x_0=0$

$$\left(x-x+\frac{x^3}{3!}-\frac{x^5}{5!}+o(x^5)\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\right)=$$

$$=\frac{x^4}{3!}-\frac{x^5}{3!\cdot 2}+\frac{x^6}{3!\cdot 3}-\frac{x^6}{5!}+o(x^6)=\frac{x^4}{3!}-\frac{x^5}{12}+\frac{17}{360}x^6+o(x^6)$$

- $f(x)=\sin x^2-\sin^2 x$ $\quad n=6$, $x_0=0$

$$\sin x^2=x^2-\frac{(x^2)^3}{6}+o((x^2)^3)=x^2-\frac{x^6}{6}+o(x^6)$$

$$\sin^2 x=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^5)\right)^2=x^2+\frac{x^6}{36}-\frac{x^4}{3}+\frac{2}{5!}x^6+o(x^6)$$

$$P_6(x)=\cancel{x^2}-\frac{x^6}{6}+o(x^6)-\cancel{x^2}-\frac{x^6}{36}+\frac{x^4}{3}-\frac{x^6}{60}+o(x^6)=\frac{x^4}{3}-\frac{19}{90}x^6+o(x^6)$$

- $f(x)=1-\sin x+x^3-\left(\frac{\pi}{2}\right)^3$ $\quad n=3$, $x_0=\pi/2$

$$t=x-\frac{\pi}{2}$$

$$f(t)=1-\sin\left(t+\frac{\pi}{2}\right)+\left(t+\frac{\pi}{2}\right)^3-\left(\frac{\pi}{2}\right)^3=1-\cos t+\left(t+\frac{\pi}{2}\right)^3-\left(\frac{\pi}{2}\right)^3=$$

$$=1-1+\frac{t^2}{2}+o(t^3)+t^3+\cancel{\left(\frac{\pi}{2}\right)^3}+3t^2\cdot\frac{\pi}{2}+3t\cdot\frac{\pi^2}{4}-\cancel{\left(\frac{\pi}{2}\right)^3}=\frac{3}{4}t\pi^2+\frac{1+3\pi}{2}t^2+t^3+o(t^3)$$

$$P_3=\left(x-\frac{\pi}{2}\right)^3+\frac{1+3\pi}{2}\cdot\left(x-\frac{\pi}{2}\right)^2+\frac{3}{4}\pi^2\left(x-\frac{\pi}{2}\right)$$

- $f(x)=\sqrt[3]{\cos(3x+x^2)}$ $\quad n=3$ $\quad x_0=0$

$$\cos(3x+x^2)=1-\frac{(3x+x^2)^2}{2}+o(x^3)=1-\frac{9x^2+x^4+6x^3}{2}+o(x^3)$$

$$\sqrt[3]{\quad}=1+\frac{1}{3}\left(-\frac{9x^2+x^4+6x^3}{2}\right)+o(x^3)=1-\frac{3}{2}x^2-x^3+o(x^3)$$

- $f(x)=e^{-3x^2}-\cos(\sqrt{6}x)$ $\quad x_0=0$

$$1-3x^2+\frac{9}{2}x^4+\frac{1}{6}(-3x^2)^3+o(x^6)-\left(1-\frac{6x^2}{2}+\frac{36}{4!}x^4+o(x^4)\right)=$$

$$=\frac{9}{2}x^4+o(x^6)-\frac{3}{2}x^4+o(x^4)=3x^4+o(x^4)$$

o.i $=4$

p.p $=3x^4$

- $f(x)=x^3+\ln(1-x^3+x^k)$ $\quad k\in\mathbb{N}$ $\quad t\to 0$

$$\ln(1+x^k-x^3)=x^k-x^3-\frac{1}{2}(x^k-x^3)^2+\frac{1}{3}(x^k-x^3)^3+o((x^k-x^3)^3)$$

$$P(x)=x^k-\frac{1}{2}(x^{2k}+x^6-2x^{k+3})+o(x^4)\qquad h=\min\{2k,6,k+3\}$$

se $k>6$ $\quad$ p.p $=-\frac{1}{2}x^6$ $\qquad$ $k<6$ $\quad$ p.p $=x^k$ $\qquad$ $k=6$ $\quad$ p.p $=\frac{1}{2}x^6$

$g(x)=\cos^2 x-\sqrt{1-2x^2}+x^k$ $\quad k\in\mathbb{N}$

GLI INTEGRALI

RICERCA DI PRIMITIVE

$F(x)$ è primitiva di $f(x) \overset{\text{def}}{\Longleftrightarrow} F'(x) = f(x)$

· se $F(x)$ è primitiva di $f(x)$, allora anche $G(x) = F(x) + c$ lo è

· $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x} \underset{\uparrow}{=} \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx = \operatorname{tg} x - \operatorname{cotg} x + c$

per la relazione fondamentale della trigonometria

$$D \operatorname{tg} x = D \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$D \operatorname{cotg} x = D \frac{\cos x}{\sin x} = -\frac{\cos^2 x + \sin^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

· $\displaystyle\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \int \frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}\,dx + \int \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}}\,dx =$

$$= \ln\left|\cos\frac{x}{2}\right| + \ln\left|\sin\frac{x}{2}\right| + c$$

· $\displaystyle\int \frac{dx}{x\sqrt{1-\ln^2 x}} \underset{\downarrow}{=} \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin(\ln x) + c$

$t = \ln x$
$dt = \frac{1}{x}dx$

· $\displaystyle\int x(1+e^{x^2})\,dx = \frac{1}{2}\int 2x(1+e^{x^2})\,dx \underset{\downarrow}{=} \frac{1}{2}\int (1+e^t)\,dt = \frac{1}{2}t + \frac{1}{2}e^t + c = \frac{x^2}{2} + \frac{e^{x^2}}{2} + c$

$t = x^2$
$dt = 2x\,dx$

· $\displaystyle\int \frac{1+2^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{1+2^{\sqrt{x}}}{2\sqrt{x}}\,dx \underset{\downarrow}{=} 2\int (1+2^t)\,dt = 2\int 1\,dt + 2\int 2^t\,dt =$

$t = \sqrt{x}$
$dt = \frac{dx}{2\sqrt{x}}$

$2^t = e^{t\ln 2}$
$D2^t = De^{t\ln 2} = \ln 2 \cdot 2^t$

$$= 2t + \frac{2}{\ln 2}\int 2^t \ln 2\,dt = 2t + \frac{2}{\ln 2}\cdot 2^t + c = 2\sqrt{x} + \frac{2}{\ln 2}2^{\sqrt{x}} + c$$

· $\displaystyle\int \frac{dx}{x\sqrt{2x-1}} = \int \frac{2\,dt}{t^2+1} = 2\operatorname{arctg} t + c = 2\operatorname{arctg}\sqrt{2x-1} + c$

$t = \sqrt{2x-1}$ $\quad t^2 = 2x-1$ $\quad x = \frac{t^2+1}{2}$
$dt = \frac{dx}{\sqrt{2x-1}}$

$$Q(z)=\frac{N(z)}{D(z)} \to \text{razionale in } z \qquad \text{es: } \frac{z^2+1}{z^3+z+3}$$

$$Q(e^x)=\frac{e^{2x}+1}{e^{3x}+e^x+3} \longrightarrow \text{razionale in } e^x$$

$$\text{allora } \int Q(e^x)\,dx \longrightarrow t=e^x$$

- $$\int \frac{e^x}{1+e^{2x}}\,dx \underset{\substack{t=e^x\\ dt=e^x dx}}{=} \int \frac{\cancel{t}}{1+t^2}\cdot\frac{dt}{\cancel{t}} = \operatorname{arctg} t + c = \operatorname{arctg} e^x + c$$

- $$\int \frac{e^x-1}{e^x+1}\,dx = \int \frac{t-1}{t+1}\,\frac{dt}{t} = \int \frac{t-1}{t(t+1)}\,dt =$$

$$\frac{t-1}{t(t+1)} = \frac{A}{t}+\frac{B}{t+1} = \frac{At+A+Bt}{t(t+1)} = \frac{(A+B)t+A}{t(t+1)} \quad \begin{cases} A+B=1 \\ A=-1 \end{cases} \begin{cases} A=-1 \\ B=2 \end{cases}$$

$$=\int -\frac{1}{t}\,dt + \int \frac{2}{t+1}\,dt = -\ln|t| + 2\ln|t+1| + c = -\ln e^x + 2\ln(1+e^x)+c =$$

$$= -x + 2\ln(1+e^x)+c$$

$$\int R(\sin x \cos x)\,dx \longrightarrow t=\operatorname{tg}\frac{x}{2} \quad \sin x=\frac{2t}{1+t^2} \quad \cos x=\frac{1-t^2}{1+t^2}$$

razionale

$$dt=\left(1+\operatorname{tg}^2\frac{x}{2}\right)\cdot\frac{1}{2}\,dx \Longrightarrow dx=\frac{2}{1+t^2}\,dt$$

- $$\int \frac{dx}{\cos x+1} = \int \frac{1}{\frac{1-t^2}{1+t^2}+1}\cdot\frac{2dt}{1+t^2} = \int \frac{\cancel{1+t^2}}{1-\cancel{t^2}+1+\cancel{t^2}}\cdot\frac{\cancel{2}dt}{\cancel{1+t^2}} = \int dt = t+c = \operatorname{tg}\frac{x}{2}+c$$

- $$\int \frac{1}{\cos x}\,dx = \int \frac{1}{\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{2}{1-t^2}\,dt$$

- $$\int \frac{\sin x}{\cos x+3\sin x}\,dx = \int \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}+3\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{4t}{(-t^2+6t+1)(1+t^2)}\,dt$$

$$\int R(\operatorname{tg} x)\,dx \qquad t=\operatorname{tg} x$$

razionali

$$dt=(1+\operatorname{tg}^2 x)\,dx \quad dx=\frac{dt}{1+t^2}$$

- $$\int \frac{2\operatorname{tg} x}{1+3\operatorname{tg} x+2\operatorname{tg}^2 x}\,dx = I \qquad I=\int \frac{2t}{(1+3t+2t^2)}\cdot\frac{dt}{(1+t^2)} =$$

$$\sin^2 x = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{tg^2 x}{tg^2 x + 1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{1}{1+tg^2 x} = \frac{1}{1+t^2}$$

$$\sin x \cos x = \frac{\sin x \cos x}{\sin^2 x + \cos^2 x} = \frac{tg x}{tg^2 x + 1} = \frac{t}{t^2+1}$$

- $$\int \frac{2\, dx}{3\cos^2 x - 2\sin^2 x + \sin x \cos x + 3} = \int \frac{2\,dx}{\frac{3}{1+tg^2 x} - \frac{2tg^2 x}{1+tg^2 x} + \frac{tg x}{1+tg^2 x} + 3} =$$

$$= \int \frac{2(1+tg^2 x)}{tg^2 x + tg x + 6}\, dx = \int \frac{2\cancel{(1+t^2)}}{t^2+t+6} \cdot \frac{dt}{\cancel{1+t^2}} = \ldots$$

INTEGRALE PER PARTI

$$\int f \cdot Dg = f \cdot g - \int g \cdot Df$$

fattor finito

fattor differenziale

- $$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$$
- $$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + c$$
- $$\int \operatorname{arctg}\sqrt{x}\, dx \underset{t=\sqrt{x}}{=} \int 2t \cdot \operatorname{arctg} t\, dt = t^2 \operatorname{arctg} t - \int \frac{t^2+1-1}{1+t^2}\, dt = t^2 \operatorname{arctg} t - \int dt + \int \frac{dt}{1+t^2} =$$

$$= t^2 \operatorname{arctg} t - t + \operatorname{arctg} t + c = x \operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x} + c$$

- $$\int |\sin x| \cos x\, dx = \sin x |\sin x| - \int \frac{|\sin x| \cos x}{\sin x} \cdot \sin x\, dx = \sin x |\sin x| - \int |\sin x| \cos x$$

$$\int |\sin x| \cos x = \frac{1}{2} \sin x |\sin x| + c$$

- $$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx = -e^x \cos x + \sin x \cdot e^x - \int e^x \sin x\, dx$$

$$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + c$$

- $$\int \sin^3 x\, dx = \int \sin x (1 - \cos^2 x)\, dx = \int \sin x\, dx - \int \sin x \cos^2 x\, dx = -\cos x + \frac{\cos^3 x}{3} + c$$

FUNZIONI ~~RAZIONALI~~ FRATTE

$$\int \frac{N_n(x)}{D_m(x)}$$

n e m grado di numeratore e denom.
$n<m$

- scomporre in fattori il denominatore

$$D_m(x) = a(x-x_1)^{m_1}(x-x_2)^{m_2} \dots (x-x_k)^{m_k}(x^2+a_1x+b_1)^{r_1}(x^2+a_2x+b_2)^{r_2}\dots (x^2+a_hx+b_h)^{r_h}$$

dove $\sum_{i=1}^{k} m_i + \sum_{i=1}^{h} 2r_i = m$

- decomporre in fratti semplici

$$\frac{N_n(x)}{D_m(x)} = \frac{A_1}{(x-x_0)^1} + \frac{A_2}{(x-x_0)^2} + \dots \frac{A^t}{(x-x_0)^t} + \frac{B_1}{(x-x_1)^1} + \frac{B_2}{(x-x_1)^2} + \dots + \frac{B^r}{(x-x_1)^r} + \dots$$

$$\dots + \frac{C_1x+D_1}{x^2+ax+b} + \frac{C_2x+D_2}{x^2+dx+e}$$

$\Delta<0 \qquad \Delta<0$

polo = 0 del denominatore

- $$\frac{ax+2}{x^3(x+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{B_1}{(x+1)} + \frac{B_2}{(x+1)^2}$$

polo 0 di molteplicità 3

polo -1 di molteplicità 2

- $$\frac{5+3x^2}{(x-1)^3(x-2)(x+5)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{B_1}{x-2} + \frac{C_1}{x+5} + \frac{C_2}{(x+5)^2}$$

polo 1 moltepl. 3

polo 2 " 1

polo -5 " 2

- $$\frac{a+x^3}{x^2(x+1)(x^2+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x+1} + \frac{C_1x+C_2}{x^2+1}$$

$m=5$ polo reale 0 molt 2

" " -1 " 1

" complesso coniugato $\pm i$

- $$\frac{5+x^4}{(x+1)^2(x^2-9)(x^2+8)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{B_1}{x-3} + \frac{C_1}{x+3} + \frac{D_1x+E_1}{x^2+8}$$

$m=6$ polo reale -1 moltep 2

" +3 e -3 molt 1

- $\int \frac{dx}{(x-x_0)} = \ln|x-x_0|+c$

- $\int \frac{dx}{(x-x_0)^m} = \frac{\Delta}{(x-x_0)^{m-1}} \cdot \frac{1}{1-m} + c$

$m>1,\ m\in\mathbb{N}$

- $\int \frac{2x+a}{x^2+ax+b}dx = \ln(x^2+ax+b)+c$

- $\int \frac{A}{x^2+ax+b}dx =$

$$\int \frac{dx}{x^2+9} = \frac{1}{9}\int\frac{dx}{\frac{x^2}{9}+1} = \frac{1}{9}\int\frac{dx}{(\frac{x}{3})^2+1} \underset{\substack{t=\frac{x}{3}\\ dt=\frac{1}{3}dx}}{=} \frac{1}{9}\int\frac{3dt}{t^2+1} = \frac{1}{3}\int\frac{dt}{t^2+1} = \frac{1}{3}\operatorname{arctg}t + c = \frac{1}{3}\operatorname{arctg}\left(\frac{x}{3}\right)+c$$

$$\int\frac{dx}{5x^2+16} = \frac{1}{16}\int\frac{dx}{(\frac{\sqrt5}{4}x)^2+1} \underset{t=\frac{\sqrt5}{4}x}{=} \frac{1}{16}\int\frac{\frac{4}{\sqrt5}dt}{t^2+1} = \frac{1}{4\sqrt5}\operatorname{arctg}\left(\frac{\sqrt5}{4}x\right)+c$$

$$\int\frac{dx}{x^2+x+1} = \int\frac{dx}{(x+\frac12)^2+\frac34} = \frac43\int\frac{dx}{\frac43(x+\frac12)^2+1} = \frac43\int\frac{dx}{\left[\frac{2}{\sqrt3}(x+\frac12)\right]^2+1} = \frac{4}{3}\int\frac{\frac{\sqrt3}{2}dt}{t^2+1} = \frac{2\sqrt3}{3}\operatorname{arctg}\left(\frac{2}{\sqrt3}x+\frac{1}{\sqrt3}\right)$$

$\Delta<0$

$(x+\alpha)^2+\beta^2 = x^2+\underbrace{2\alpha x}+\alpha^2+\beta^2$

$2\alpha=1$, $\alpha=\frac12$, $\beta^2=\frac34$

$t=\frac{2}{\sqrt3}(x+\frac12)$

$dt=\frac{2}{\sqrt3}dx$

$$\int\frac{dx}{x^2+2x+5} = \int\frac{dx}{(x+1)^2+4} = \frac14\int\frac{dx}{\frac14(x+1)^2+1} = \frac14\int\frac{dx}{(\frac{x}{2}+\frac12)^2+1} = \frac14\int\frac{2dt}{t^2+1} = \frac12\operatorname{arctg}\left(\frac{x}{2}+\frac12\right)+c$$

$\Delta<0$

$x^2+2x+5=(x+1)^2+4$

$t=\frac{x}{2}+\frac12$

$dt=\frac12 dx$

- $\int\frac{x+1}{x^3+x}dx$

chiedere a Jessica

$\frac{x+1}{x(x^2+1)} = \frac{A}{x}+\frac{Bx+C}{x^2+1}$

$A=\lim_{x\to0}\left(\frac{x+1}{x^2+1}\right)=1$

$\frac{Bx+C}{x^2+1} = \frac{x+1}{x(x^2+1)}-\frac1x = \frac{x+1-x^2-1}{x(x^2+1)} = \frac{\cancel{x}(1-x)}{\cancel{x}(x^2+1)} =$

$A=1$
$B=-1$
$C=1$

$$\int\frac1x dx+\int\frac{-x+1}{x^2+1} = \int\frac1x dx-\frac12\int\frac{2x}{x^2+1}dx+\int\frac{1}{x^2+1}dx = \ln|x|-\frac12\ln(x^2+1)+\operatorname{arctg}x+c$$

- $\displaystyle\int \frac{3x-1}{x^3-5x^2+8x-6}dx = \int \frac{3x-1}{(x-1)(x-2)^2}dx = 2\int\frac{1}{x-1}dx + 5\int\frac{1}{(x-2)^2} + -2\int\frac{1}{x-2} = 2\ln|x-1| - 2\ln|x-2| - \frac{5}{x-2}$

$x^3-5x^2+8x-6 = (x-1)(x^2-6x+6) =$

$$\begin{array}{c|ccc|c} & 1 & -5 & +8 & -6 \\ 1 & & 1 & -6 & +6 \\ \hline & 1 & -6 & +6 & 0 \end{array}$$

$= (x-1)(x-2)^2$

$$\frac{3x-1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

$A = \lim_{x\to 1}\frac{3x-1}{(x-2)^2} = 2$

$B = \lim_{x\to 2}\left(\frac{3x-1}{x-1}\right) = 5$

$C = -2$

$$\frac{C}{x-2} = \frac{3x-1}{(x-1)(x-2)^2} - \frac{2}{x-1} - \frac{5}{(x-2)^2} = \ldots = \frac{-2\cancel{(x^2-3x+2)}}{\cancel{(x-1)}(x-2)^{\cancel{2}}} = \frac{-2}{x-2}$$

$(x-1)(x-2)$

$$\int\frac{3x-1}{x^3-5x^2+8x-6}dx = 2\ln|x-1| - 2\ln|x-2| - 5(x-2)^{-1}$$

- $\displaystyle\int\frac{6x^2+1}{(x-1)(x^2-6)}dx = -\frac{5}{3}\ln|x-1| + \frac{17}{6}\ln|x-2| + \frac{17}{12}\ln|x+2| + C$

$$\frac{4x^2+1}{(x-1)(x-2)(x+2)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+2}$$

$A = \lim_{x\to 1}\frac{4x^2+1}{\underbrace{(x-2)(x+2)}_{x^2-6}} = -\frac{5}{3}$

$B = \lim_{x\to 2}\frac{6x^2+1}{(x-1)(x+2)} = \frac{17}{1\cdot 6} = \frac{17}{6}$

$C = \lim_{x\to -2}\frac{6x^2+1}{(x-1)(x-2)} = \frac{17}{-3\cdot(-6)} = \frac{17}{12}$

- $\displaystyle\int\frac{x+2}{x^2+3x+15}dx = \frac{1}{2}\int\frac{2x+6}{x^2+3x+15}dx = \frac{1}{2}\int\frac{2x+3}{x^2+3x+15} + \frac{1}{2}\int\frac{1}{x^2+3x+15} = \frac{1}{2}\ln|x^2+3x+15| + \frac{\sqrt{51}}{51}\operatorname{arctg}\left(\frac{2}{\sqrt{51}}x+\frac{3}{\sqrt{51}}\right)$

$x^2+3x+15 = (x+\frac{3}{2})^2 + \frac{51}{6}$

$$\frac{6}{51}\int\frac{1}{\frac{6}{51}(x+\frac{3}{2})^2+1}dx = \frac{6}{51}\int\frac{1}{(\frac{2}{\sqrt{51}}x+\frac{3}{\sqrt{51}})^2+1}dx = \frac{6}{51}\int\frac{\frac{\sqrt{51}}{2}dt}{t^2+1} = \frac{2\sqrt{51}}{51}\operatorname{arctg} t$$

$t = \frac{2}{\sqrt{51}}x + \frac{3}{\sqrt{51}}$

$dt = \frac{2}{\sqrt{51}}dx$

$$\int \frac{x^5+2x^4-x+7}{x^3+3x^2+5x+3}\,dx = \int x^2-x-2\,dx + \int \frac{8x^2+12x+13}{x^3+3x^2+5x+3}\,dx = \frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{9}{2}\ln|x+1| +$$

$$+\frac{7}{4}\ln|x^2+2x+3| -$$

$$-2\sqrt{2}\,\text{arctg}\left(\frac{x}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)+c$$

$$\begin{array}{l|l}
x^5+2x^4-x+7 & x^3+3x^2+5x+3 \\
-x^5-3x^4-5x^3-3x^2 & \\ \hline
/\ -x^4-5x^3-3x^2-x+7 & x^2-x-2 \\
+x^4+3x^3+5x^2+3x & \\ \hline
/\ -2x^3+2x^2+2x+7 & \\
+2x^3+6x^2+10x+6 & \\ \hline
/\ 8x^2+12x+13 &
\end{array}$$

$$x^3+3x^2+5x+3 = (x+1)(x^2+2x+3)$$

$$\begin{array}{c|ccc|c}
 & 1 & 3 & 5 & 3 \\
-1 & & -1 & -2 & -3 \\ \hline
 & 1 & +2 & 3 & 0
\end{array}$$

$$\frac{8x^2+12x+13}{(x+1)(x^2+2x+3)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+3}$$

$$A = \lim_{x\to -1}\frac{8x^2+12x+13}{x^2+2x+3} = \frac{8+13-12}{1-2+3} = \frac{9}{2}$$

$$\frac{Bx+C}{x^2+2x+3} = \frac{8x^2+12x+13}{(x+1)(x^2+2x+3)} - \frac{9}{2(x+1)} = \frac{16x^2+24x+26-9x^2-18x-27}{2(x+1)(x^2+2x+3)} = \frac{\overbrace{7x^2+6x-1}^{(x+1)(7x-1)}}{2(x+1)(x^2+2x+3)}$$

$$\frac{Bx+C}{x^2+2x+3} = \frac{7x-1}{2(x^2+2x+3)} = \frac{\frac{7}{2}x-\frac{1}{2}}{x^2+2x+3} \qquad B=\frac{7}{2} \quad C=-\frac{1}{2}$$

$$\frac{9}{2}\int\frac{1}{x+1}\,dx + \frac{7}{2}\int\frac{x-1/7}{x^2+2x+3}$$

$$\frac{7}{4}\int\frac{2x-2/7}{x^2+2x+3} = \frac{7}{4}\int\frac{2x+2}{x^2+2x+3} - \frac{7}{4}\cdot\frac{16}{7}\int\frac{1}{x^2+2x+3}$$

$$-4\int\frac{1}{x^2+2x+3}\,dx = -4\cdot\frac{1}{2}\int\frac{1}{\left(\frac{x}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)^2+1}\,dx = -2\int\frac{\sqrt{2}}{t^2+1}\,dt = -2\sqrt{2}\,\text{arctg}\, t$$

$$(x+1)^2+2 \qquad t=\frac{x}{\sqrt{2}}+\frac{1}{\sqrt{2}} \qquad dt=\frac{1}{\sqrt{2}}\,dx$$

INTEGRALI IMPROPRI

$$\cdot \int_{-1}^{1}\frac{dx}{\sqrt{|x|}(x-4)} = \overbrace{\int_{-1}^{0}\frac{dx}{\sqrt{-x}(x-4)}}^{I_a} + \overbrace{\int_{0}^{1}\frac{dx}{\sqrt{x}(x-4)}}^{I_b} = -\text{arctg}\frac{1}{2}-\frac{1}{2}\ln 3$$

$\boxed{x=0}$ $f(x) = \frac{1}{\sqrt{|x|}\cdot(x-4)} \sim -\frac{1}{4}\frac{1}{\sqrt{|x|}} = \frac{-\frac{1}{4}}{|x|^{1/2}}$ infinito di ordine $\frac{1}{2}<1 \Rightarrow$ converge

$$I_a = \int_{-1}^{0}\frac{dx}{\sqrt{-x}(x-4)} = \lim_{c\to 0^-}\int_{-1}^{c}\frac{dx}{\sqrt{-x}(x-4)} = \lim_{c\to 0^-}\left[\text{arctg}\frac{\sqrt{-x}}{2}\right]_{-1}^{c} = \lim_{c\to 0^-}\left(\text{arctg}\frac{\sqrt{-c}}{2}-\text{arctg}\frac{1}{2}\right) = -\text{arctg}\frac{1}{2}$$

$$t=\sqrt{-x} \qquad dt=\frac{1}{2\sqrt{-x}}\cdot -dx \qquad \int\frac{-2\,dt}{-t^2-4} = +2\int\frac{1}{4(t^2/4+1)} = \frac{1}{2}\cdot 2\,\text{arctg}\left(\frac{\sqrt{-x}}{2}\right)+k$$

$$I_b=\int_0^1 \frac{dx}{\sqrt{x}(x-4)}=\lim_{b\to 0^+}\int_c^1 \frac{dx}{\sqrt{x}(x-4)}=\lim_{b\to 0^+}\left[\frac{1}{2}\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right|\right]_c^1=\lim_{b\to 0^+}\left(\frac{1}{2}\ln\frac{1}{3}-\frac{1}{2}\ln\left|\frac{\sqrt{c}-2}{\sqrt{c}+2}\right|\right)=-\frac{1}{2}\ln 3$$

$$\int\frac{dx}{\sqrt{x}(x-4)}=\int\frac{2dt}{t^2-4}=\cancel{\dots}=\int\frac{1/2}{t-2}dt+\int\frac{1/2}{t+2}dt=\frac{1}{2}\ln|\sqrt{x}-2|-\frac{1}{2}\ln|\sqrt{x}+2|$$

$t=\sqrt{x}$

$dt=\frac{1}{2\sqrt{x}}dx$

$$\frac{2}{t^2-4}=\frac{A}{t-2}+\frac{B}{t+2}$$

$$A=\lim_{t\to 2}\frac{2}{t+2}=\frac{1}{2}$$

$$B=\lim_{t\to -2}\frac{2}{t-2}=-\frac{1}{2}$$

- $\displaystyle\int_0^1\frac{(e^x-e)\sqrt{1-x^2}}{x\ln^3 x}dx$ valutare se converge o diverge

$\boxed{x\to 0^+}$ $\quad f(x)=\dfrac{(e^x-e)\sqrt{1-x^2}}{x\ln^3 x}\sim\dfrac{(1-e)\cdot 1}{x\ln^3 x}$

$e^x-e\sim 1-e$

$$\int_0^{1/2}\frac{(1-e)}{x\ln^3 x}dx=\lim_{\varepsilon\to 0^+}\int_\varepsilon^{1/2}\frac{1-e}{x\ln^3 x}dx=\lim_{\varepsilon\to 0^+}\left[\frac{e-1}{2\ln^2 x}\right]_\varepsilon^{1/2}=\frac{e-1}{2\ln^2\frac{1}{2}}-\underbrace{\frac{e-1}{2\ln^2\varepsilon}}_{\to 0}$$

$$\int\frac{1-e}{x\ln^3 x}=\dots=-\frac{1-e}{2\ln^2 x}+c \qquad =\frac{e-1}{2\ln^2\frac{1}{2}} \text{ converge}$$

Per $x\to 0^+$ $f(x)$ converge

$\boxed{x\to 1}$

$$f(x)=\frac{(e^x-e)\sqrt{1-x^2}}{x\ln^3 x}=\frac{(e^{1+t}-e)\sqrt{1-(1+t)^2}}{(1+t)\ln^3(1+t)}=\frac{e(e^t-1)\sqrt{-2t-t^2}}{(1+t)\ln^3(1+t)}=$$

$t=x-1\to 0^-$ $\qquad e^t-1=1+t+o(t)-1=t+o(t)$

$$\sqrt{-2t-t^2}=(-2t)^{1/2}\sqrt{1+\frac{t}{2}}=(-2t)^{1/2}\left(1+\frac{t}{4}+o(t)\right)=$$
$$=(-2t)^{1/2}+o\left(-t^{1/2}\right)$$

$$[\ln(1+t)]^3=[t+o(t)]^3=t^3+o(t^3)$$

$$f(x)\sim\frac{e\cdot t\cdot(-2t)^{1/2}}{1\cdot t^3}=\frac{e\sqrt{2}(-t)^{1/2}}{(-t)^2}=\frac{e\sqrt{2}}{(-t)^{3/2}}=\frac{e\sqrt{2}}{(1-x)^{3/2}}$$

ordine di infinito $\frac{3}{2}$ $\leadsto$ diverge

$\Downarrow$

diverge

$$\int_1^{+\infty}\left[\frac{1}{x^k}-\sin\left(\frac{1}{x}\right)\right]dx = \lim_{\varepsilon\to+\infty}\int_1^{\varepsilon}\frac{1}{x^k}-\sin\left(\frac{1}{x}\right)$$

$$f(x)=\frac{1}{x^k}-\sin\left(\frac{1}{x}\right) = t^k - \cancel{\sin} t + \frac{t^3}{6}+o(t^3) \sim \begin{cases} -t & k>1 \\ t^3/6 & k=1 \\ t^k & k<1\end{cases}$$

$\to$ diverge
$\to$ converge
$\to$ diverge

$t=\frac{1}{x}\to 0^+$

$\sin t_{t\to 0} = t-\frac{t^3}{6}+o(t^3)$

ordine di infinitesimo

è un integrale convergente se $k=1$

EQUAZIONI DIFFERENZIALI

A VARIABILI SEPARABILI

- $y'=f(x)\cdot g(y)$

$$\begin{cases}\text{dom} f(x)\\ \text{dom}\, g(y)\end{cases}$$

- $g(y)=0$ $\to y=k_1$, $y=k_2$ $\quad k_1, k_2 \in \mathbb{R}$ $\to$ integrale singolare

- $(x-3)^2 y' = x(y-1) \qquad x\neq 3$

$y'=\frac{x}{(x-3)^2}(y-1) \qquad x\neq 3$

- $f(x)=\frac{x}{(x-3)^2} \qquad \text{dom} f = \{\forall x\in\mathbb{R} \mid x\neq 3\}$
- $g(y)=y-1 \qquad \text{dom}\, g = \{\forall y\in\mathbb{R}\}$

- $g(y)=y-1=0 \qquad y=1$ con $x\neq 3$ integrale singolare dell'equazione differenziale

- Separazione delle variabili

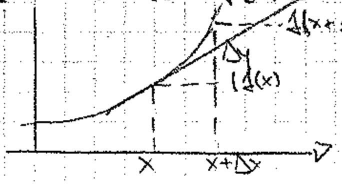
$$\frac{dy}{dx}=\frac{x}{(x-3)^2}(y-1)$$

$$\int\frac{dy}{y-1}=\int\frac{x}{(x-3)^2}dx$$

- $\int\frac{dy}{y-1}=\ln|y-1|+c$
- $\int\frac{x}{(x-3)^2}=\frac{1}{2}\int\frac{2x-6+6}{(x-3)^2}dx=\frac{1}{2}\int\frac{2x-6}{x^2-6x+9}dx + 3\int\frac{1}{(x-3)^2}dx=\frac{1}{2}\ln(x-3)^2-\frac{3}{x-3}+c$

$dy=f'(x)dx \qquad f'(x)=\frac{dy}{dx}$

$f(x+\Delta x)-f(x)=f'(x)\Delta x+o(\Delta x)$



- $xy' = y(1+y^2) \Rightarrow y' = \frac{1}{x} \cdot y(1+y^2)$

  - $\text{dom} f = \{\forall x \in \mathbb{R} : x \neq 0\}$

    $\text{dom}\, g = \{\forall y \in \mathbb{R}\}$

  - $g(y) = y(1+y^2) = 0 \quad y = 0$ integrale singolare

  - $\int \frac{dy}{y(1+y^2)} = \int \frac{dx}{x}$

$$\int \frac{dy}{y(1+y^2)} = \int \frac{1}{y} dy - \int \frac{y}{1+y^2} dy = \ln|y| - \frac{1}{2}\ln(1+y^2) + c$$

$$\frac{1}{y(1+y^2)} = \frac{A}{y} + \frac{By+C}{1+y^2}$$

$$A = \lim_{y \to 0} \frac{1}{1+y^2} = 1 \qquad \frac{By+C}{1+y^2} = \frac{\cancel{1}-\cancel{1}-y^2}{y(1+y^2)} = -\frac{y}{1+y^2} \quad \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

$$\int \frac{1}{x} dx = \ln|x| + c_1$$

$$\ln|y| - \frac{1}{2}\ln(1+y^2) = \ln|x| + k \qquad k = \ln k_0 \quad k_0 > 0$$

$$\ln \frac{|y|}{\sqrt{1+y^2}} = \ln\left[|x| \cdot k_0\right]$$

$$\frac{|y|}{\sqrt{1+y^2}} = k_0|x| \quad , k_0 \in \mathbb{R}$$

$$\frac{y}{\sqrt{1+y^2}} = \bar{k}_0|x| \quad , \quad \bar{k}_0 \neq 0 \text{ e } y = 0$$

$$\frac{y}{\sqrt{1+y^2}} = hx \quad , \ h \in \mathbb{R}$$

$$\frac{y^2}{1+y^2} = h^2x^2$$

$$y^2 = h^2x^2 + h^2x^2y^2$$

$$y^2 = \frac{h^2x^2}{1-h^2x^2}$$

$$y = \pm |h| \frac{x}{\sqrt{1-h^2x^2}}$$

- $\begin{cases} (1+e^x)yy' = e^x \\ y(0) = 1 \end{cases} \rightsquigarrow P(0,1)$

$$y' = \frac{e^x}{1+e^x} \cdot \frac{1}{y}$$

  - $\text{dom} f = \{\forall x \in \mathbb{R}\}$

    $\text{dom}\, g = \{\forall y \in \mathbb{R} : y \neq 0\}$

NON CI SONO INTEGRALI SINGOLARI

  - $\int y\,dy = \int \frac{e^x}{1+e^x} dx \qquad \int y\,dy = \frac{y^2}{2} + c_1$

$$\begin{cases} \frac{1}{2}y^2 = \ln(1+e^x) + c \quad c \in \mathbb{R} \\ y(0) = 1 \end{cases} \qquad \int \frac{e^x}{1+e^x} dx = \ln(1+e^x) + c_2$$

SOLUZIONE DEL PROBLEMA DI CAUCHY

$$\frac{1}{2} = \ln(2) + c \qquad c = \frac{1}{2} - \ln 2 \qquad \frac{1}{2}y^2 = \ln(1+e^x) + \frac{1}{2} - \ln 2$$

• $y' = y|x-1|$

$$\int \frac{dy}{y} = \int |x-1|\,dx$$

$$-\int \frac{dy}{y} = \ln|y| + c_1$$

$$-\int |x-1|\,dx \begin{cases} \text{se } x \geq 1 & \int (x-1)\,dx = \frac{x^2}{2} - x + c_2 \\ \text{se } x<1 & \int (1-x)\,dx = -\frac{x^2}{2} + x + c_3 \end{cases}$$

Impongo la continuità in $x=1$

$$\lim_{x\to 1^-} \left(-\frac{x^2}{2} + x + c_3\right) = -\frac{1}{2} + 1 + c_3 = c_3 + \frac{1}{2}$$

$$\lim_{x\to 1^+} \left(\frac{x^2}{2} - x + c_2\right) = \frac{1}{2} - 1 + c_2 = c_2 - \frac{1}{2}$$

$$F(1) = c_2 - \frac{1}{2} \qquad c_3 + \frac{1}{2} = c_2 - \frac{1}{2} \qquad c_2 = c_3 + 1$$

$$\int |x-1|\,dx = \begin{cases} \frac{x^2}{2} - x + c_3 + 1, & x \geq 1 \\ -\frac{x^2}{2} + x + c_3, & x<1 \end{cases}$$

$$\ln|y| + c_1 = \begin{cases} \frac{x^2}{2} - x + c_3 + 1 & x \geq 1 \\ -\frac{x^2}{2} + x + c_3 & x<1 \end{cases} \qquad c_3 - c_1 = k \in \mathbb{R}$$

$$\ln|y| = \begin{cases} \frac{x^2}{2} - x + k + 1 & x \geq 1 \\ -\frac{x^2}{2} + x + k & x<1 \end{cases}$$

• $\begin{cases} y' = |y^2-1| \cdot x \\ y(0) = -2 \end{cases}$ $\quad P(0,-2)$

$$|y^2-1| = \begin{cases} y^2-1 & y \leq -1 \vee y \geq 1 \\ 1-y^2 & -1<y<1 \end{cases}$$

• $g(y)=0 \quad y=\pm 1$

$\Updownarrow$

$$\begin{cases} y' = (y^2-1)x \\ y(0) = -2 \end{cases}$$

$$\int \frac{dy}{y^2-1} = \int x\,dx$$

$$\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1} \qquad A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\int \frac{dy}{y^2-1} = \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| + c_1 \quad c_1 \in \mathbb{R}$$

$$\int x\,dx = \frac{x^2}{2} + c_2 \qquad c_2 \in \mathbb{R}$$

$$\cancel{\frac{1}{2}} \ln\left|\frac{y-1}{y+1}\right| = \frac{x^2}{\cancel{2}} + k$$

$$\left|\frac{y-1}{y+1}\right| = e^{x^2+k} \qquad \left|\frac{y-1}{y+1}\right| = k_0 \cdot e^{x^2} \quad k_0 \in \mathbb{R}^+$$

$$\frac{y-1}{y+1} = \bar{k_0} \cdot e^{x^2} \qquad \bar{k_0} \neq 0$$

$$y - 1 = \bar{k_0} e^{x^2} y + \bar{k_0} e^{x^2}$$

$$y(1 - \bar{k_0} e^{x^2}) = 1 + \bar{k_0} e^{x^2} \qquad \bar{k_0} \in \mathbb{R} \setminus \{0\}$$

$$\begin{cases} y = \frac{1 + \bar{k_0} e^{x^2}}{1 - \bar{k_0} e^{x^2}} \\ y(0) = -2 \end{cases} \qquad -2 = \frac{1 + \bar{k_0}}{1 - \bar{k_0}} \qquad -2 + 2\bar{k_0} = 1 + \bar{k_0} \quad \bar{k_0} = 3$$

$$\boxed{y = \frac{1 + 3e^{x^2}}{1 - 3e^{x^2}}}$$

OMOGENEE

$$\boxed{y' = f\left(\frac{y}{x}\right) \qquad z = \frac{y}{x} \rightsquigarrow y = x \cdot z \qquad y' = z + x \cdot z'}$$

- $\begin{cases} y' = \dfrac{x^2+y^2}{xy} \\ y(1) = 2 \end{cases}$

$\cancel{z} \neq 0 \quad y \neq 0$

$$y' = \frac{1+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}$$ divido sopra e sotto per x^2

$$\Downarrow$$

$$z + x \cdot z' = \frac{1+z^2}{z}$$

$$xz' = \frac{1+z^2-z^2}{z}$$

$$z' = \frac{1}{zx} \qquad \varphi(z) = \frac{1}{z} \qquad \psi(x) = \frac{1}{x}$$

$\varphi(z) = 0$ no integrali singolari

$$\int z\,dz = \int \frac{dx}{x}$$

$$\frac{z^2}{2} = \ln|x| + c$$

$$z^2 = 2\ln|x| + C_0 \qquad C_0 \in \mathbb{R}$$

$$\begin{cases} \dfrac{y^2}{x^2} = 2\ln|x| + C_0 \\ y(1) = 2 \end{cases}$$

$$4 = 2\ln 1 + C_0 \qquad C_0 = 4$$

$$y^2 = 2x^2\ln|x| + 4x^2$$

- $y' = e^{\left(\frac{y}{x}\right)} + \left(\frac{y}{x}\right)$

$$z = \frac{y}{x} \qquad x \neq 0$$

$$\cancel{z} + x \cdot z' = e^z + \cancel{z}$$

$$z' = \frac{e^z}{x}, \quad x \neq 0$$

$e^z = 0 \rightsquigarrow$ NO integrali singolari

$$\int \frac{dz}{e^z} = \int \frac{dx}{x} \qquad -e^{-z} = \ln|x| + c \qquad c \in \mathbb{R}$$

$$e^{-z} = \ln|x|^{-1} + c$$

$$-z = \ln\left(\ln\frac{1}{|x|} + c\right)$$

$$y = -x\ln\left(\ln\frac{1}{|x|} + c\right) \qquad c \in \mathbb{R}$$

LINEARI

$$\boxed{\begin{array}{c} y' + a(x)y = b(x) \qquad A(x) = \int a(x)\,dx \\ y = e^{-A(x)} \int e^{A(x)} \cdot b(x)\,dx \end{array}}$$

- $y' = xy + x^3$

$$a(x) = -x$$
$$b(x) = x^3$$
$$A(x) = \int -x\,dx = -\frac{x^2}{2} + \cancel{c}$$
$$y = e^{\frac{x^2}{2}} \int e^{-\frac{x^2}{2}} \cdot x^3\,dx$$

$$\int x^3 e^{-\frac{x^2}{2}}\,dx = \frac{1}{2}\int t\, e^{-t/2}\,dt = \frac{1}{2}\left[-2e^{-t/2}\cdot t + \int +2e^{-t/2}\,dt\right] = -e^{-t/2}\cdot t - 2e^{-t/2} + c =$$

$x^2 = t$
$2x\,dx = dt$

$$= -x^2 e^{-\frac{x^2}{2}} - 2e^{-\frac{x^2}{2}} + c$$

$$y = \cancel{e^{\frac{x^2}{2}}} \cdot (-x^2) \cdot \cancel{e^{-x^2/2}} - 2\cancel{e^{x^2/2}} \cdot \cancel{e^{-x^2/2}} + c \cdot e^{x^2/2} = -x^2 - 2 + c \cdot e^{\frac{x^2}{2}}$$

- $\begin{cases} y' + \frac{2x}{1+x^2}y = \frac{1}{x(1+x^2)} \\ y(1) = 2 \end{cases}$

$$a(x) = \frac{2x}{1+x^2} \qquad b(x) = \frac{1}{x(1+x^2)}$$
$$A(x) = \int \frac{2x}{1+x^2}\,dx = \ln(1+x^2)$$
$$y = e^{-\ln(1+x^2)} \int e^{+\ln(1+x^2)} \cdot \frac{1}{x(1+x^2)}\,dx = \frac{1}{1+x^2}\int \cancel{(1+x^2)} \cdot \frac{1}{x\cancel{(1+x^2)}}\,dx =$$
$$= \frac{1}{1+x^2}\int \frac{1}{x}\,dx = \frac{\ln|x|}{1+x^2} + \frac{c}{1+x^2}$$

$$2 = \frac{c}{2} \quad \leadsto \quad c = 4$$

$$y = \frac{\ln|x|}{1+x^2} + \frac{4}{1+x^2}$$

- $xy' - y = x^2\cos x$

$$y' - \frac{1}{x}y = x\cos x \qquad x \neq 0 \qquad a(x) = -\frac{1}{x} \qquad b(x) = x\cos x$$
$$A(x) = \int -\frac{1}{x}\,dx = -\ln|x| + c$$
$$y = e^{+\ln|x|}\int e^{-\ln|x|} \cdot x\cos x\,dx = |x| \int \frac{x}{|x|}\cos x\,dx \leadsto \begin{array}{l} x > 0 \quad y = x\int \cos x\,dx \\ x < 0 \quad y = +x\int \cos x\,dx \end{array}$$

$$y = x\sin x + \cancel{c}$$

$\circ \quad xy' = y(\ln y - \ln x)$

$y' = \frac{y}{x} \ln\left(\frac{y}{x}\right)$

$\frac{y}{x} = z \qquad x \neq 0$

$z + x \cdot z' = z \cdot \ln z$

$z' = \frac{z(\ln z - 1)}{x}$

$\int \frac{dz}{z(\ln z - 1)} = \int \frac{dx}{x} \qquad \int \frac{dx}{x} = \ln|x| + C_1$

$\int \frac{dz}{z(\ln z - 1)} = \ln(\ln z - 1) + C_2$

$\ln|x| = \ln(\ln z - 1) + K \qquad K \in \mathbb{R}$

$\ln(\ln z - 1) = \ln(|x| \cdot K_0) \qquad K_0 \in \mathbb{R}^+$

$\ln z - 1 = |x| \cdot K_0 \qquad K_0 \in \mathbb{R}$

$z = e^{|x| \cdot K_0 + 1}$

$z = e^{x \cdot \overline{K} + 1} \qquad \overline{K} \neq 0$

$y = x \cdot e^{x\overline{K} + 1} \qquad \overline{K} \neq 0$