CAMPI ELETTROMAGNETICI

Equazioni di Maxwell

$$\begin{cases} \text{rot}\,\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \text{rot}\,\vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \\ \text{div}\,\vec{D} = \rho_s \end{cases}$$

$\Rightarrow \text{div}(B) = 0 \Rightarrow \text{rot}\,\vec{A} = \vec{B}$ ($\vec{A}$: potenziale vettore)

La novità rispetto ad elettrotecnica II è il $\dfrac{\partial \vec{D}}{\partial t}$.

$$\text{rot}\,\vec{E} = -\frac{\partial}{\partial t}\,\text{rot}\,\vec{A} \Rightarrow \text{rot}\,\vec{E} = -\text{rot}\,\frac{\partial \vec{A}}{\partial t} \Rightarrow \text{rot}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\text{grad}\,V \Rightarrow \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \text{grad}\,V$$

(V: potenziale scalare)

$$\text{rot}\,\vec{H} = J + \frac{\partial D}{\partial t}$$

Nel quasi stazionario:

$$\text{rot}(\vec{H}) = \text{rot}\left(\frac{B}{\mu}\right) = J = \sigma E = \text{rot}\left(\frac{1}{\mu}\,\text{rot}\,\vec{A}\right) =$$

$$\frac{1}{\mu}\left(\underbrace{\text{grad}\,\text{div}\,\vec{A}}_{=0} - \vec{\nabla}^2\vec{A}\right) = \sigma\left(-\frac{\partial A}{\partial t} - \text{grad}\,V\right)$$

$=0$ per condizione di Coulomb

Una delle ipotesi è $\vec{J} = \vec{J}_s$ } sorgente impressa
l'altra ipotesi è $\rho = \rho_s$ }

L'equazione diventa $-\vec{\nabla}^2\vec{A} = \vec{J}_s$

$$\text{div}\,\varepsilon E = \rho_s \qquad \text{div}\,\vec{E} = \frac{\rho_s}{\varepsilon}$$

$$\text{div}\left(-\frac{\partial A}{\partial t} - \text{grad}\,V\right) = \frac{\rho_s}{\varepsilon}$$

$$-\frac{\partial}{\partial t}\underbrace{\text{div}\,A}_{=0} - \nabla^2 V = \frac{\rho_s}{\varepsilon} \Rightarrow -\nabla^2 V = \frac{\rho_s}{\varepsilon}$$

$$\vec{A} = \frac{\mu}{4\pi}\int_{\text{volume}} \frac{\vec{J}_s}{r}\,d\tau \qquad V = \frac{1}{4\pi\varepsilon}\int_{\text{vol.}} \frac{\rho_s}{r}\,d\tau$$

Nel caso non stazionario

J e ρ sono sorgenti del campo e $\text{rot}\, \vec{H} = \vec{J}_s - \frac{\partial \vec{D}}{\partial t}$

$$\text{rot}\left(\frac{1}{\mu}\,\text{rot}(\vec{A})\right) = J_s + \varepsilon \frac{\partial E}{\partial t} = \vec{J}_s + \varepsilon \frac{\partial}{\partial t}\left(-\frac{\partial \vec{A}}{\partial t} - \text{grad}\, V\right)$$

$$= \vec{J}_s - \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} - \varepsilon\, \text{grad} \frac{\partial V}{\partial t}$$

$$\text{grad}(\text{div}\,\vec{A}) - \vec{\nabla}^2 \vec{A} = \mu\left[\vec{J}_s + \varepsilon \frac{\partial^2 \vec{A}}{\partial t} - \varepsilon\, \text{grad} \frac{\partial V}{\partial t}\right]$$

il $\text{grad}(\text{div}\,\vec{A})$ e $\text{grad} \frac{\partial V}{\partial t}$ danno fastidio.

Impongo la condizione di Lorentz tale che:

$$\text{div}\,\vec{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$$

Imponendo la condizione di Lorentz ottengo

$$-\vec{\nabla}^2 \vec{A} = \mu \vec{J}_s - \mu\varepsilon \frac{\partial^2 A}{\partial t}$$

$$\boxed{-\vec{\nabla}^2 \vec{A} + \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = \mu \vec{J}_s}$$

$$\text{div}\,\vec{D} = \rho_s \Rightarrow \text{div}\,(\varepsilon \vec{E}) = \rho_s$$

$$\text{div}\left(-\frac{\partial A}{\partial t} - \text{grad}\, V\right) = \frac{\rho_s}{\varepsilon}$$

$$-\text{div} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}^2 V = \frac{\rho_s}{\varepsilon}$$

Applichiamo la condizione di Lorentz $\text{div}\, A = -\mu\varepsilon \frac{\partial V}{\partial t}$

$$\mu\varepsilon \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho_s}{\varepsilon} \Rightarrow \boxed{-\nabla^2 V + \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = \frac{\rho_s}{\varepsilon}}$$

$$V = \frac{1}{4\pi\varepsilon} \int_{\text{volume}} \frac{\rho_s\left(t - \frac{r}{a}\right)}{r}\, d\tau$$

↑ potenziale scalare ritardato

$$a = \frac{1}{\sqrt{\mu\varepsilon}} = 3\cdot 10^8 \frac{1}{\sqrt{\mu_r \varepsilon_r}}$$

se $\mu = \mu_0 = 4\pi \cdot 10^{-7}\ \frac{H}{m}$

$\varepsilon = \varepsilon_0 = \frac{1}{36\pi} \cdot 10^{-9}$

$$a = \frac{1}{\sqrt{4\pi \cdot 10^{-7} \cdot \frac{1}{36\pi} \cdot 10^{-9}}} = 3\cdot 10^8\ \frac{m}{s}$$

$$\vec{A} = \frac{\mu}{4\pi} \int_{\text{vol.}} \frac{J_s\left(t - \frac{r}{a}\right)}{r}\, d\tau$$

↑ potenziale vettore ritardato

il ritardo dipende da r e dalla velocità a

Se siamo in regime sinusoidale

$V(t) \Rightarrow \bar{V}$

$\vec{A}(t) \Rightarrow \vec{A} \begin{cases} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{cases}$

$\frac{\partial}{\partial t} \rightarrow j\omega$

Quindi $\begin{cases} -\nabla^2 \bar{V} + \omega^2 \mu \varepsilon \bar{V} = \frac{\rho_s}{\varepsilon} \\ -\nabla^2 \vec{A} - \omega^2 \mu \varepsilon \vec{A} = \mu \bar{J}_s \end{cases}$ $\quad V = \frac{1}{4\pi\varepsilon} \int \frac{\rho_s e^{-j\frac{r}{c}\omega}}{r} d\tau$

$\frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} = K$ numero d'onda $\Rightarrow e^{-j\frac{r\omega}{c}} = e^{-jkr}$

definiamo $\lambda = \frac{c}{f} = cT$

Nel vuoto alla freq di 50 Hz,

$c = 3 \cdot 10^8 \quad \lambda = \frac{3 \cdot 10^8}{50} = 6 \cdot 10^6$ m $= 6000$ km

se $f = 1$ GHz $\quad \lambda = \frac{3 \cdot 10^8}{10^9} = 0{,}3$ m

Nota: ad alte frequenze con distanze piccole si possono avere sfasamenti importanti (e^{-jkr} se f grande $\rightarrow \lambda$ piccolo $\rightarrow K$ grande).

<u>Campi elettromagnetici con andamento sinusoidale in regioni prive di sorgenti</u>

$$\begin{cases} rot(\vec{E}) = -j\omega\mu\vec{H} \\ rot(\vec{H}) = \vec{J}_s + j\omega\varepsilon\vec{E} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = \underbrace{(\sigma + j\omega\varepsilon)}_{j\omega\hat{\varepsilon}}\vec{E} \\ div(\vec{D}) = 0 \quad (\rho_s = 0) \end{cases}$$

$$\hat{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$$

se applichiamo $div(rot\,\vec{E}) \Rightarrow div(\vec{H}) = 0$

se facciamo $div(rot\,H) \Rightarrow div(j\omega\hat{\varepsilon}\vec{E}) = 0 \Rightarrow div(\vec{E}) = 0$

$rot(rot\,\vec{E}) = -j\omega\mu\, rot(H) = -j\omega\mu(j\omega\hat{\varepsilon})\vec{E} = \omega^2\mu\hat{\varepsilon}\vec{E}$

$grad(\cancel{div\,\vec{E}}) - \nabla^2\vec{E}$ (con $div\,\vec{E} = 0$)

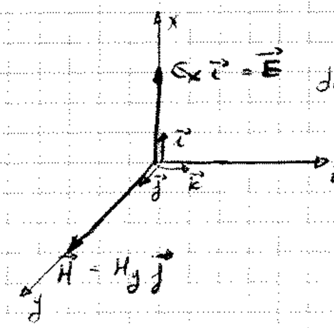
$$-\nabla^2\vec{E} = \underbrace{\omega^2\mu\hat{\varepsilon}}_{-\gamma^2}\vec{E}$$

facendo la stessa operazione

$$\text{rot}(\text{rot}\,\vec{H}) = \text{rot}(j\omega\hat{\varepsilon}\vec{E}) = \text{rot}(\vec{E})\,j\omega\hat{\varepsilon} = \omega^2\mu\hat{\varepsilon}\vec{H}$$

$$\text{grad}(\underbrace{\text{div}\,\vec{H}}_{=0}) - \nabla^2\vec{H} = \omega^2\mu\hat{\varepsilon}\vec{H} \Rightarrow -\nabla^2\vec{H} = \underbrace{\omega^2\mu\hat{\varepsilon}}_{-\gamma^2 = -(m+jn)}\vec{H}$$

Caso delle onde piane uniformi



$\text{div}\,\vec{D} = 0 \rightarrow \text{div}\,\vec{E} = 0$

$\text{div}\,\vec{B} = 0 \rightarrow \text{div}\,\vec{H} = 0$

$$\begin{cases} \text{rot}(\vec{H}) = j\omega\hat{\varepsilon}\vec{E} \\ \text{rot}(\vec{E}) = -j\omega\hat{\mu}\vec{H} \end{cases}$$

$$\text{rot}(H) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = j\omega\hat{\varepsilon}\vec{E}$$

$$\text{rot}(H) = -\frac{\partial H_y}{\partial z}\vec{i} + \underbrace{\frac{\partial H_y}{\partial x}}_{=0}\vec{k} = -\frac{\partial H_y}{\partial z}\vec{i} = j\omega\hat{\varepsilon}E_x\vec{i}$$

$$\text{rot}\,\vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\vec{j} - \underbrace{\frac{\partial E_x}{\partial y}}_{=0}\vec{k} = \frac{\partial E_x}{\partial z}\vec{j} = -j\omega\mu H_y\vec{j}$$

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y$$

$$\begin{cases} -\frac{\partial H_y}{\partial z} = j\omega\hat{\varepsilon}E_x \\ +\frac{\partial E_x}{\partial z} = j\omega\mu H_y \end{cases} \Rightarrow \begin{cases} -\frac{\partial^2 H_y}{\partial z^2} = j\omega\hat{\varepsilon}\frac{\partial E_x}{\partial z} \\ +\frac{\partial^2 E_x}{\partial z^2} = j\omega\mu\frac{\partial H_y}{\partial z} \end{cases} \Rightarrow \begin{cases} -\frac{\partial^2 H_y}{\partial z^2} = +\omega^2\mu\hat{\varepsilon}H_y \\ +\frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu\hat{\varepsilon}E_x \end{cases} \Rightarrow$$

$$\begin{cases} +\frac{\partial^2 H_y}{\partial z^2} = \gamma^2 H_y \Rightarrow \\ +\frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x \Rightarrow \end{cases} \begin{cases} H_y = H_y^+ e^{-\gamma z} + H_y^- e^{\gamma z} \\ E_x = E_x^+ e^{-\gamma z} + E_x^- e^{\gamma z} \end{cases}$$

$$E_x = -\frac{1}{j\omega\hat{\varepsilon}}\frac{d}{dz}\left[H_y^+ e^{-\gamma z} + H_y^- e^{\gamma z}\right] = -\frac{1}{j\omega\hat{\varepsilon}}\left[-\gamma H_y^+ e^{-\gamma z} + \gamma H_y^- e^{\gamma z}\right] =$$

$$= \underbrace{\frac{\gamma}{j\omega\hat{\varepsilon}} H_y^+}_{E_x^+} e^{-\gamma z} + \underbrace{\frac{\gamma}{j\omega\hat{\varepsilon}} H_y^-}_{E_x^-} e^{\gamma z}$$

Da cui si ottiene che

$$\frac{E_x^+}{H_y^+} = \frac{\gamma}{j\omega\hat{\varepsilon}} = \frac{\sqrt{-\omega^2\mu\hat{\varepsilon}}}{j\omega\hat{\varepsilon}} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = +\eta$$ — impedenza caratteristica del mezzo

$$\frac{E_x^-}{H_y^-} = -\eta$$

$$H_y = -\frac{1}{j\omega\mu}\left[-\gamma E^{+} e^{-\gamma z} + \gamma E^{-} e^{\gamma z}\right] = \underbrace{\frac{\gamma}{j\omega\mu} E^{+}}_{H^{+}} e^{-\gamma z} - \underbrace{\frac{\gamma}{j\omega\mu} E^{-}}_{H^{-}} e^{\gamma z}$$

$$\frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu\, j\omega\hat{\varepsilon}}} = \sqrt{\frac{j\omega\mu}{j\omega\hat{\varepsilon}}} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = Z_{\infty} = \eta$$

$$\boxed{\frac{E^{+}}{H^{+}} = \eta \qquad \frac{E^{-}}{H^{-}} = -\eta}$$

Campi di applicazione:

- isolante perfetto $(\sigma = 0)$
- conduttore perfetto $(\sigma \to \infty)$

$$\begin{cases} \gamma^2 = -\omega^2\mu \overbrace{\left(\varepsilon - j\frac{\sigma}{\omega}\right)}^{\hat{\varepsilon}} = m + jn \\ \eta = \sqrt{\dfrac{\mu}{\varepsilon + j\frac{\sigma}{\omega}}} \\ \gamma = \alpha + j\beta \quad \to \gamma^2 = \alpha^2 - \beta^2 + j\,2\alpha\beta \end{cases}$$

$$\begin{cases} m = \alpha^2 - \beta^2 \\ n = 2\alpha\beta \end{cases} \Rightarrow \begin{cases} m^2 = \alpha^4 + \beta^4 - 2\alpha^2\beta^2 \\ n^2 = 4\alpha^2\beta^2 \end{cases} \qquad (m^2+n^2) = (\alpha^2+\beta^2)^2$$

$$\Downarrow$$

$$\alpha^2 + \beta^2 = \sqrt{m^2+n^2}$$

Quindi otteniamo:

$$\alpha = \sqrt{\frac{\sqrt{m^2+n^2}+m}{2}} \qquad \beta = \sqrt{\frac{\sqrt{m^2+n^2}-m}{2}}$$

Si ottiene che

$$\alpha = \sqrt{\frac{\omega^2\mu\sqrt{\varepsilon^2+\sigma^2} - \omega^2\mu\varepsilon}{2}} \qquad \beta = \sqrt{\frac{\omega^2\mu\sqrt{\varepsilon^2+\sigma^2} + \omega^2\mu\varepsilon}{2}}$$

$$e^{+}(t) = \sqrt{2}\,|E_x^{+}|\,\sin(\omega t + \varphi_+) \quad \text{per } z = 0$$

$$\left.\begin{aligned} e^{+}(t) &= \sqrt{2}\,|E_x^{+}|\,\sin(\omega t + \varphi_+ - \beta z)\,e^{-\alpha z} \\ e^{-}(t) &= \sqrt{2}\,|E_x^{-}|\,e^{+\alpha z}\sin(\omega t + \varphi_- + \beta z) \end{aligned}\right\} \Rightarrow e(t) = e^{+}(t) + e^{-}(t)$$

$$h_y(t,z) = \sqrt{2}\,\frac{|E_x^{+}|}{|\eta|}\,e^{-\alpha z}\sin(\omega t + \varphi_+ - \beta z - \varphi_\eta) - \sqrt{2}\,\frac{|E_x^{-}|}{|\eta|}\,e^{\alpha z}\sin(\omega t + \varphi_- + \beta z - \varphi_\eta)$$

mezzi senza perdite ($\sigma = 0$) (aria, vuoto)

$\alpha = 0 \qquad \beta = \omega\sqrt{\mu\varepsilon}$

$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0$

La velocità di propagazione risulta $c = \frac{1}{\sqrt{\mu\varepsilon}}$ (vedi slide per capire il perché)

$\lambda = \frac{\omega T}{\beta} = \frac{2\pi}{T}\cdot\frac{T}{\beta} = \frac{2\pi}{\beta}$ lunghezza d'onda.

$\lambda = c \cdot T \qquad \lambda = \Delta z \rightarrow$ spazio percorso in un periodo

Quindi nel caso senza perdite: ($c_0 \rightarrow 3\cdot 10^8\ \frac{m}{s}$)

$\eta = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} \qquad c = \frac{c_0}{\sqrt{\mu_r\varepsilon_r}}$

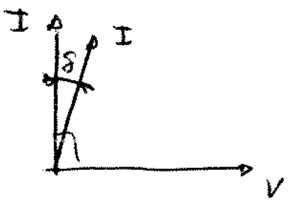
$\lambda = \frac{2\pi}{\beta} = \frac{c}{f} = \frac{c_0}{f\sqrt{\mu_r\varepsilon_r}}$

Mezzi con perdite

$\hat{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$ definiamo $\tan\delta = \frac{\sigma}{\omega\varepsilon}$

un buon conduttore ha $\frac{\sigma}{\omega\varepsilon} \gg 1 \Leftrightarrow \frac{\omega\varepsilon}{\sigma} \rightarrow 0$

un buon ~~isolante~~ dielettrico ha $\frac{\sigma}{\omega\varepsilon} \ll 1 \Leftrightarrow$ ~~$\frac{\omega\varepsilon}{\sigma}$~~ $\frac{\sigma}{\omega\varepsilon} \rightarrow 0$



condensatore ideale sfasa corrente di 90°, uno non ideale lo sfasa di $90° - \delta$

$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{j\omega\mu\sigma\left(1 + j\frac{\omega\varepsilon}{\sigma}\right)}$

$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$ spessore di penetrazione

se $\sigma \rightarrow \infty$ allora $\gamma \simeq \sqrt{j\omega\mu\sigma}$

$\gamma \simeq \sqrt{\frac{\omega\mu\sigma}{2}}(1+j) = \frac{1}{\delta}(1+j)$

$\eta = \sqrt{\frac{j\omega\mu}{\sigma\left(1 + j\frac{\omega\varepsilon}{\sigma}\right)}} = \sqrt{\frac{\omega\mu}{\sigma}}\frac{(1+j)}{\sqrt{2}} = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$

Ragionamenti energetici:

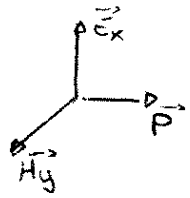
$\vec{E} = E_x \vec{i}$ $\vec{H} = H_y \vec{j}$

$\vec{E_x} \times \vec{H_y} = \vec{P} = \vec{P}_k$ $P = E_x H_y^*$

prodotto vettoriale — vettore di Poynting

Sapendo che

$$\begin{cases} E_x = \left(E^+ e^{-\alpha t} e^{-j\beta z} + E^- e^{\alpha t} e^{j\beta z}\right) \\ H_y = \left(\frac{E^+}{\eta} e^{-\alpha t} e^{-j\beta z} - \frac{E^-}{\eta} e^{\alpha t} e^{j\beta z}\right) \end{cases}$$



$$P = E_x H_y^* = \left[E^+ e^{-\alpha t} e^{-j\beta z} + E^- e^{\alpha t} e^{j\beta z}\right]\left[\frac{E^{+*}}{\eta^*} e^{-\alpha t} e^{j\beta z} - \frac{E^{-*}}{\eta^*} e^{\alpha t} e^{-j\beta z}\right]$$

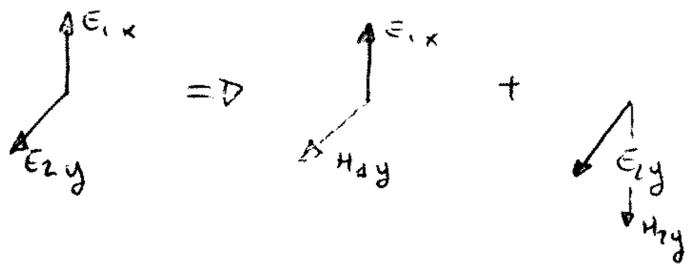
$$P = \frac{E^+ E^{+*}}{\eta^*} e^{-2\alpha t} + \frac{E^- E^{+*}}{\eta^*} e^{2j\beta z} - \frac{E^+ E^{-*}}{\eta^*} e^{-2j\beta z} - \frac{E^- E^{-*}}{\eta^*} e^{2\alpha t}$$

$$= \frac{|E^+|^2}{\eta^*} e^{-2\alpha t} - \frac{|E^-|^2}{\eta^*} e^{2\alpha t} + \frac{|E^+||E^-|}{\eta^*} e^{j\varphi_-} e^{-j\varphi_+} e^{2j\beta z} - \frac{|E^+||E^-|}{\eta^*} e^{j\varphi_+} e^{-j\varphi_-} e^{-2j\beta z}$$

$$= \frac{1}{\eta^*}\left\{\underbrace{|E^+|^2 e^{-2\alpha t} - |E^-|^2 e^{2\alpha t}}_{\text{Parte reale}} + \underbrace{|E^+||E^-| \underbrace{\left[e^{j\varphi_-} e^{-j\varphi_+} e^{2j\beta z} - e^{j\varphi_+} e^{-j\varphi_-} e^{2j\beta z}\right]}_{-2j\sin(\varphi_+ - \varphi_- - 2\beta z)}}_{\text{Parte immaginaria}}\right\}$$

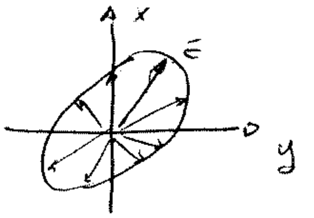
Polarizzazione di un'onda: se sovrappongo ad un campo E_1 un altro campo E_2 e ad H_1 sovrappongo H_2. Supponiamo che entrambi i campi siano sullo stesso piano $x-y$.

Il campo elettromagnetico sarà polarizzato in modo diverso a seconda di come si sovrappongono i due campi.



$E_{1x} = \sqrt{2}\,|E_{1x}|\, e^{-\alpha t} \sin(\omega t + \varphi_{1+} - \beta z) + \dots$

$E_{2y} = \sqrt{2}\,|E_{2y}|\, e^{-\alpha t} \sin(\omega t + \varphi_2 - \beta z) + \dots$



se E_{1x} ed E_{2y} sono sfasati di angolo $\neq 0$ e da $\frac{\pi}{2}$ ottengo un campo rotante che si muove lungo un ellissoide.

La composizione sarà:

$$\bar{E} = A \sin$$