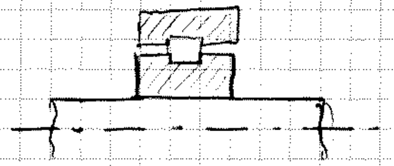
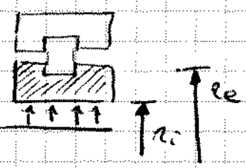
Esercizio 1



$p_i = 13 \text{ MPa}$



$\sigma_{re} = 0$

$\mu(\beta) = ?$

disco fermo e non soggetto a grad (T)

$$\sigma_r(X) = A - \frac{B}{X^2} \qquad \left.\begin{aligned} \sigma_r(\beta) &= -p_i \\ \sigma_r(1) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &A = B \\ &A = p_i \frac{\beta^2}{1-\beta^2} \end{aligned}$$

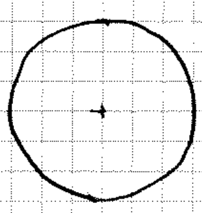
$$\varepsilon_c = \begin{cases} \dfrac{u}{r} = \dfrac{1}{r_e} \dfrac{u(X)}{X} \\[2ex] \dfrac{1}{E}(\sigma_c - \nu \sigma_r) \quad \text{NOTA !!} \end{cases}$$

$$\varepsilon_c(\beta) = \frac{1}{E}\left[\sigma_c(\beta) - \nu \sigma_r(\beta)\right] \quad \text{NOTA}$$

$$\mu(\beta) = \varepsilon_c(\beta)\, r_e\, \beta = 8{,}2\ \mu m$$

Esercizio 2

Disco non forato rotante. Per effetto della sola rotazione



$$\mu(1) = 0{,}25 \text{ mm}$$

$$d_e = 310 \text{ mm} \qquad \omega = ?$$

$$E = 2 \cdot 10^5 \text{ MPa} \qquad \nu = 0{,}3 \qquad \rho = 7800 \ \frac{kg}{m^3}$$

$$\sigma_r = A - \frac{B}{X^2} - \sigma_0 \frac{3+\nu}{8} X^2, \qquad \sigma_0 = \rho \omega^2 r_e^2$$

Poiché disco pieno $B = 0$

$$\sigma_r(1) = 0 \Rightarrow A = \sigma_0 \frac{3+\nu}{8}$$

$$\varepsilon_c = \frac{u}{r} \Rightarrow \mu(1) = 0{,}25 \text{ mm} = r_e\, \varepsilon_c(1) \qquad \varepsilon_c = \frac{1}{E}\, \sigma_{ce} - \nu\, \sigma_{re}$$

$$\mu(1) = r_e \frac{1}{E}(\sigma_{ce} - \nu \overbrace{\cancel{\sigma_{re}}}^{=0})$$

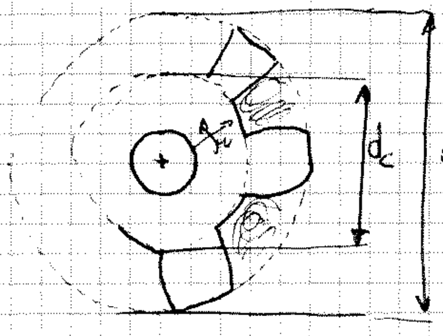
$$\sigma_{ce} = A + \underbrace{\cancel{\frac{B}{X^2}}}_{=0} - \sigma_0 \frac{1+3\nu}{8} X^2 = \frac{\sigma_0}{8}\left(3+\nu-1-3\nu\right) X^2 = \frac{\sigma_0}{4} X^2 (1-\nu)$$

$$\mu(1) = \frac{r_e}{E}\frac{\sigma_0}{4}(1-\nu) = \frac{r_e}{E}\frac{\rho\omega^2 r_e^2}{4}(1-\nu) \Rightarrow \omega = \sqrt{\frac{\mu \cdot 4E}{(1-\nu) r_e^3 \rho}} = 2440 \ \frac{rad}{s}$$

E1

Esercizio 3

Pacco di lamierini



$\omega = 1800$ rpm

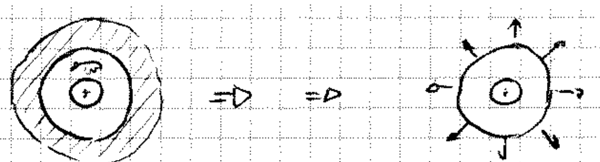
$d_i = 200$ mm

$d_c = 800$ mm $d_e = 1300$ mm

$E = 2 \cdot 10^5$ MPa $\nu = 0{,}3$

La parte resistente si limita fino a d_c.

$\sigma_{r_i} = 0$



$$\sigma_{re} = \frac{F_c}{2\pi d_c h} \qquad dF_c = dm\,\omega^2 r = \rho\, 2\pi r\, dr\, h \cdot \rho\, \omega^2\, 2\pi r^2\, dr\, h$$

$$F_c = \int_{r_c}^{r_e} dF_c = 2\pi \rho \omega^2 h \, \frac{r_e^3 - r_c^3}{3}$$

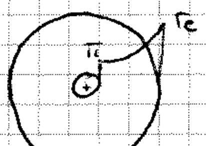
$$\sigma_{re} = \frac{\rho \omega^2 (r_e^3 - r_c^3)}{3 r_c}$$

$$\left.\begin{aligned} \sigma_r(\beta) &= 0 \\ \sigma_r(1) &= \sigma_{re} \end{aligned}\right\} \text{ Ricavo A e B}$$

$$\sigma_c(x) \ \forall x \rightarrow \sigma_c(\beta) = 160 \text{ MPa}$$

$$u(\beta) = r_i\, \varepsilon_c(\beta) = r_i \frac{1}{E}\left[\sigma_c(\beta) - \nu \underbrace{\sigma_r(\beta)}_{=0}\right]$$

Esercizio 4



$$T(x) = b_1 x + b_2 x^2 \qquad b_1 = 200°C \quad b_2 = 100°C \qquad \beta = \frac{r_i}{r_e} = 0{,}4$$

$E = 2 \cdot 10^5$ MPa $\nu = 0{,}3$ $\rho = 7800 \frac{kg}{m^3}$ $\alpha = 12 \cdot 10^{-6}\ °C^{-1}$

$$\sigma_r = A - \frac{B}{x^2} + \sum_{i=1}^{n} C_i x^i \qquad C_i = -\frac{\alpha E b_i}{i+2}$$

$$\sigma_r = A - \frac{B}{x^2} + C_1 x + C_2 x^2 \qquad \sigma_r(\beta) = 0$$

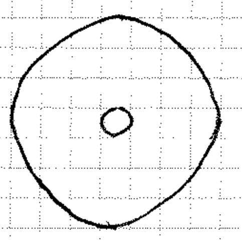
$$\sigma_r(1) = 0$$

$$\sigma_c = A + \frac{B}{x^2} + \sum (1+i)\, c_i\, x^i$$

Quindi sia $\sigma_r(x)$ che $\sigma_c(x)$ sono note.

Sicuramente $\underset{265\ \text{MPa}}{\sigma_c(\beta)} > 0$ e $\underset{-224\ \text{MPa}}{\sigma_c(1)} < 0$

Esercizio 5



$R_m = 905\ \text{MPa}$ $\quad R_{p0,2} = 635\ \text{MPa}$ $\quad \nu = 0{,}3$

$\rho = 7800\ \frac{kg}{m^3}$

$d_i = 100$ $\quad d_e = 300$

$\omega_{cp} \cong 13160$ rpm

la condizione limite si ha quando tutto il disco cavo è con sollecitazione R_s. Questa velocità è ω_{cp}: velocità di collasso plastico.

$\omega_{cp} = 18667$ rpm NON resiste a 20000 rpm

se voglio avere $\omega_{cp} = 20200$ rpm. Impongo ω_{cp} e ricavo

R_s. $R_s^* = 840\ \text{MPa}$

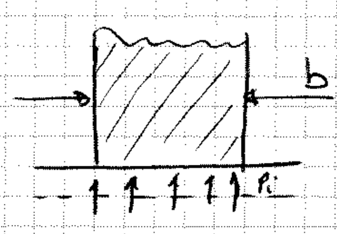
Esercizio su ε_z

Ricordiamo che $\quad \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r - \sigma_c)\right]$

$E = 1{,}1 \cdot 10^5\ \text{MPa}$ $\quad \nu = 0{,}25$ $\quad b = 75$ mm $\quad p_i = 100\ \text{MPa}$

DOPO FORZAMENTO CON p_i

$\delta = 0{,}04$ mm $\Rightarrow b \to b - \delta$



$\sigma_c(\beta) = ?$

$\sigma_r(\cancel{x}) = A - \frac{B}{x^2}$ $\qquad \sigma_c = A + \frac{B}{x^2}$

$\sigma_r + \sigma_c = 2A$ = costante su tutto il disco

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_c) = -2\frac{\nu}{E}A = \frac{-\delta}{b} \Rightarrow A = 117\ \text{MPa}$$

$$\begin{cases} \sigma_r(\beta) + \sigma_c(\beta) = 2A \implies \sigma_c(\beta) = 2A + p_i = 334 \text{ MPa} \\ \sigma_r(\beta) = -p_i \end{cases}$$

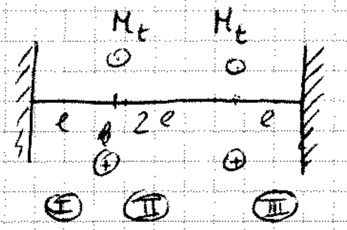
domanda extra: quanto vale β del nostro mozzo

$$\sigma_r(1) = 0 \implies A - B = 0 \implies B = A = 117 \text{ MPa}$$

$$\sigma_r(\beta) = -p_i \implies A + \frac{B}{\beta^2} = -p_i \implies \beta = \sqrt{\frac{B}{A + p_i}}$$

08/11/11

Esercizio 1



Consideriamo 3 elementi trave di torsione.

$$\begin{Bmatrix} M_1 \\ M_{2I} \end{Bmatrix} = \frac{G J_p}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \qquad \begin{matrix} \theta_1 = 0 \\ \theta_4 = 0 \end{matrix}$$

$$\begin{Bmatrix} M_{2II} \\ M_{3II} \end{Bmatrix} = \frac{G J_p}{2l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$\begin{Bmatrix} M_{3III} \\ M_4 \end{Bmatrix} = \frac{G J_p}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_3 \\ \theta_4 \end{Bmatrix}$$

Componiamo i 3 sistemi:

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3/2 & -1/2 & 0 \\ 0 & -1/2 & 3/2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \frac{G J_p}{l} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}, \qquad \begin{cases} \theta_1 = 0 \\ \theta_4 = 0 \end{cases}$$

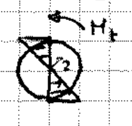
$$\begin{Bmatrix} M_2 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} M_t \\ M_t \end{Bmatrix} = \frac{G J_p}{2l} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

da cui si ottiene che $\theta_2 = +\theta_3 = \dfrac{M_t l}{G J_p}$

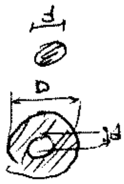
Poiché $\theta_2 = \theta_3$ mi attendo che la trave 2 sia scarica.

Dalle equazioni si ottiene ancora che $M_1 = -\dfrac{G J_p}{l} \theta_2 = -M_t$ e

$M_4 = -G \dfrac{J_p}{l} \theta_3 = -M_t$



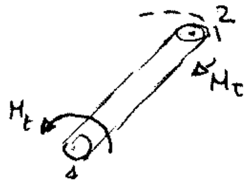
$$\tau(r) = \frac{M_t}{J_p} r \implies \tau_{max} = \frac{M_t}{J_p} \frac{d}{2}, \quad W_t = \frac{J_p}{d/2}$$



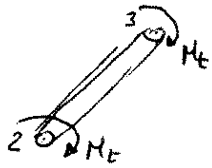
$$J_p = \frac{\pi d^4}{32} \Rightarrow W_t = \frac{\pi d^3}{16}$$

$$J_p = \frac{\pi}{32}(D^4 - d^4) \Rightarrow W_t = \frac{\pi}{16}\,\frac{D^4 - d^4}{D}$$

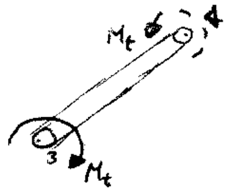
$$\tau_1 = \frac{M_1}{W_t} = -\frac{M_t}{W_t} \qquad \tau_{2I} = \frac{M_{2I}}{W_t} = \frac{M_t}{W_t}$$ elemento I



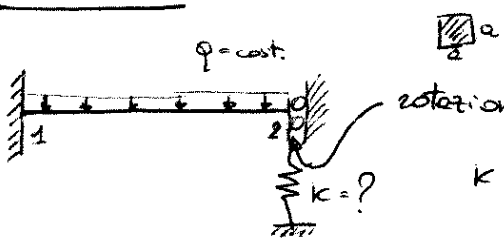
$$\left.\begin{array}{l} \tau_{2II} = \dfrac{M_{2II}}{W_t} = 0 \ , \quad M_{2II} = \dfrac{G J_p}{2\ell}(\theta_2 - \theta_3) = 0 \\ \tau_{3II} = \dfrac{M_{3II}}{W_t} = 0 \end{array}\right\}$$ elemento II



$$\tau_{3III} = \frac{M_{3III}}{W_t} = \frac{M_t}{W_t} \ , \qquad \tau_4 = \frac{M_4}{W_t} = -\frac{M_t}{W_t}$$ elemento III



Esercizio 2:



rotazione impedita, è possibile lo scorrimento verticale

k tale che $v_{2\,max} = 5\,mm$

$$q = 100\ N/mm \qquad \ell = 1\,m \qquad E = 2\cdot 10^5\ MPa \qquad a = 10\,mm$$

Equilibrio

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} + \begin{Bmatrix} \frac{q\ell}{2} \\ \frac{q\ell^2}{12} \\ \frac{q\ell}{2} \\ -\frac{q\ell^2}{12} \end{Bmatrix} = \frac{EJ}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 + k\frac{\ell^3}{EJ} & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{Bmatrix} , \qquad \begin{array}{l} v_1 = 0 \\ \varphi_1 = 0 \\ \varphi_2 = 0 \end{array}$$

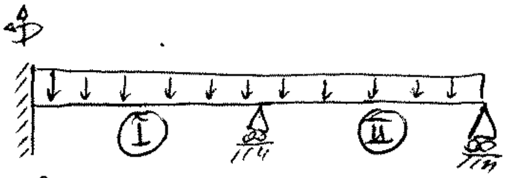
$$F_2 + q\frac{\ell}{2} = \frac{EJ}{\ell^3}\left(12 + k\frac{\ell^3}{EJ}\right) v_2 \ , \qquad F_2 = 0 \quad e \quad v_2 = 5\,mm$$

$$q\frac{\ell}{2} = \left(12\frac{EJ}{\ell^3} + k\right) v_2 \Rightarrow k = \frac{q\frac{\ell}{2} - 12\frac{EJ}{\ell^3} v_2}{v_2} =$$

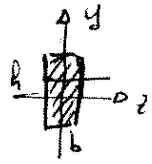
$$J = \frac{a^4}{12} \ , \quad q = -100\ N/mm \Rightarrow k_{min} \cong 10^4\ \frac{N}{mm}$$

Esercizio 3: per conto nostro rifare esercizio 2 ma ~~con~~ senza il vincolo di rotazione. $(\varphi_2 \neq 0)$

Eserzizio 4:



$q = cost$



$b = 20\ mm$
$h = 40\ mm$
$\ell = 12\ m$
$q = 600\ N/m$

$E = 2\cdot10^5\ Pa$

la trave è iperstatica

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_{2I} \\ M_{2I} \end{Bmatrix} + \begin{Bmatrix} q\ell/2 \\ q\ell^2/12 \\ q\ell/2 \\ q\ell^2/12 \end{Bmatrix} = \frac{EJ}{\ell^3}\Big[{}^{4\times4}\Big]\begin{Bmatrix} v_1 \\ \varphi_1 \\ v_{2I} \\ \varphi_{2I} \end{Bmatrix}$$

$$\begin{Bmatrix} F_{2II} \\ M_{2II} \\ F_3 \\ M_3 \end{Bmatrix} + \begin{Bmatrix} q\ell/2 \\ q\ell^2/12 \\ q\ell/2 \\ -q\ell^2/12 \end{Bmatrix} = \frac{EJ}{\ell^3}\Big[\quad\Big]\begin{Bmatrix} v_{2II} \\ \varphi_{2II} \\ v_1 \\ \varphi_1 \end{Bmatrix}$$

Assembliamo e otteniamo:

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} + \begin{Bmatrix} q\ell/2 \\ q\ell^2/12 \\ q\ell \\ 0 \\ q\ell/2 \\ -q\ell^2/12 \end{Bmatrix} = \frac{EJ}{\ell^3}\overset{6\times6}{\begin{bmatrix} & & 0 \\ & \begin{matrix} 24 & 0 \\ 0 & 8\ell^2 \end{matrix} & \\ 0 & & \end{bmatrix}}\begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{Bmatrix}$$

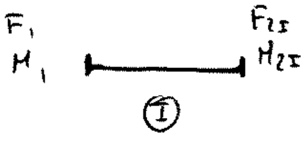
$v_1 = 0$
$\varphi_1 = 0$
$v_2 = 0$
$v_3 = 0$

$$\begin{Bmatrix} M_2 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -q\ell^2/12 \end{Bmatrix} = \frac{EJ}{\ell^3}\begin{bmatrix} 8\ell^2 & 2\ell^2 \\ 2\ell^2 & 4\ell^2 \end{bmatrix}\begin{Bmatrix} \varphi_2 \\ \varphi_3 \end{Bmatrix} \quad , M_2 = M_3 = 0$$

$$\varphi_2 = \frac{q\ell^3}{168EJ} \qquad \varphi_3 = -4\varphi_2$$

$$F_1 = \frac{13}{28}q\ell \quad , \quad M_1 = \frac{q\ell^3}{14} \quad , \quad F_2 = \frac{8}{7}q\ell \qquad F_3 = \frac{11}{28}q\ell$$

elemento I



$$\sigma_{max} = \frac{M_f}{W_f} = \frac{M_f}{J_f}y_{max} \qquad J_f = \frac{bh^3}{12} \quad y_{max} = \frac{h}{2}$$

$$\sigma_{1,max} = \frac{M_1}{W_f}$$

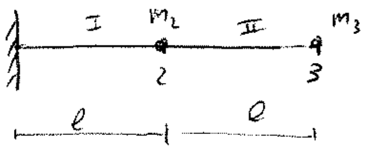
$$\sigma_{2,max} = \frac{M_{2I}}{W_f}$$

elemento II

$$\sigma_{2II,max} = \frac{M_{2II}}{W_f}$$

$$\sigma_3 = \frac{M_3}{W_f} = 0$$

13/12/11 Esercitazione



$K_I = K_2 = K = \frac{EA}{\ell}$

$m_2 = m_3 = m = 10$ Kg

$$\left([K] - \lambda^2 [M]\right)\{U\} = \{0\}$$

$$\begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}$$

Dobbiamo imporre $\left|[K] - \lambda^2 [M]\right| = 0$

$\lambda^2_{1,2} = \frac{1}{2}\frac{K}{m}\left(3 \pm \sqrt{5}\right)$ $\qquad \lambda_1 < \lambda_2 < \ldots < \lambda_n$

prima frequenza propria

$\lambda^2_1 = \frac{1}{2}\frac{K}{m}\left(3 - \sqrt{5}\right)$!!

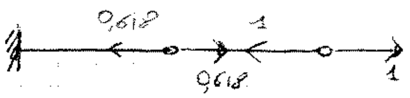
$$\underbrace{\left([K] - \lambda^2_1 [M]\right)}\{U\} = 0 \qquad \boxed{\frac{U_2}{U_3} = 0{,}618}$$

$|U_3| > |U_2|$ in quel modo di vibrare

Se $\frac{U_3}{U_2} > 0$ oscillazioni in fase, altrimenti in controfase

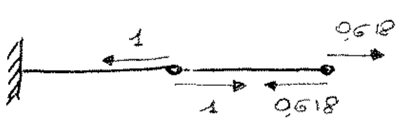
Per trovare valore di $\{U\}$ posso normalizzare*:

- $\sqrt{U_2^2 + U_3^2} = 1$ oppure
- $\{U^{(1)}\}^T [M] \{U^1\} = 1$ oppure
- $U_{MAX} = 1$



Per $\lambda^2 = \lambda^2_2$ si ha che $\frac{U_3}{U_2} = -0{,}618$ cioè si ha che

- $|U_2| > |U_3|$
- le masse oscillano in contro fase

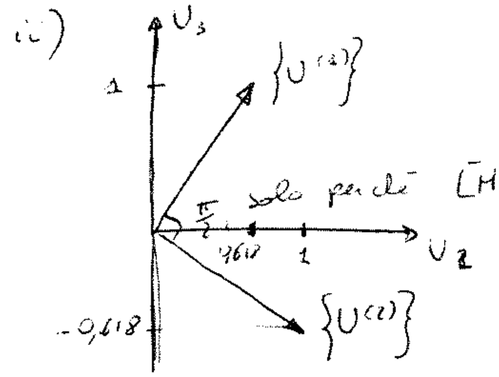


1) Verificare le proprietà di K e M ortogonalità:

2) Rappresentare graficamente nel piano dei due modi di vibrare

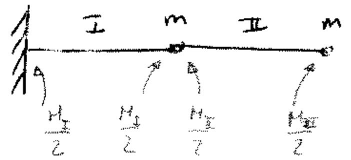
i) cioè rispetto a k $\{U^{(1)}\}^T[K]\{U^{(2)}\}=0$ idem per le masse

ii)



solo perché $[M]=m\begin{bmatrix}1&0\\0&1\end{bmatrix}=m\,\mathbb{I}$

iii) effetto delle masse dell'albero



$M_I = M_{II} = \rho A l = 0{,}38$ kg $\quad \rho = 7800$ kg/m³

$$M = m\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}\frac{M_I+M_{II}}{2}&0\\0&\frac{M_{II}}{2}\end{pmatrix}$$

in questo caso $\{U^{(2)}\}$ non è più $\perp$ a $\{U^{(1)}\}$

λ_1^2, λ_2^2 minori ~~maggiori~~ che nel caso precedente perché è aumentata la massa ($\lambda_m^2=\frac{k}{m}$ nell'oscillatore armonico)

$[M]=\begin{pmatrix}m&0\\0&m\end{pmatrix}$ $\quad$ tolto tratto 1 $\quad$ $[K]=\underbrace{\begin{bmatrix}k&-k\\-k&k\end{bmatrix}}_{\text{matrice di 1 elemento asta.}}$

$$|[K]-\lambda^2[M]|=0$$

$$\begin{bmatrix}k-\lambda^2 m & -k\\ k & k-\lambda^2 m\end{bmatrix}=0 \qquad (k-\lambda^2 m)^2-k=0$$

$\boxed{\lambda_1=0}$

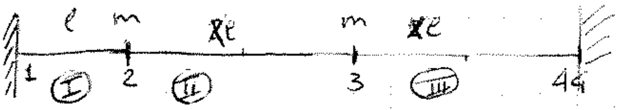
$\lambda_2=\sqrt{2}\sqrt{\frac{k}{m}}$

Se $\lambda_1=0$ vuol dire che non c'è nulla che oscilla (cioè espressione in termini dinamici del MOTO RIGIDO)

$$\begin{pmatrix}k&-k\\-k&k\end{pmatrix}\begin{Bmatrix}U_2\\U_3\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \Rightarrow U_2=U_3$$ non è un modo di vibrare!!

Un sistema non vincolato sia in statica che dinamica si muove di moto rigido!!

10/01/2012



Oscillazioni libere

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = \frac{EA}{\ell} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \left| [K] - \lambda [M] \right| = 0$$

$$m^2 \lambda^4 - 4 k m \lambda^2 + 3k^2 = 0 \qquad \lambda_1^2 = \frac{k}{m} \qquad \lambda_2^2 = 3\frac{k}{m}$$

$$\left([K] - \lambda^2 [M] \right) \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\lambda^2 = \lambda_1^2 = \frac{k}{m} \Rightarrow U_2 = U_3$$

in fase

$$\lambda^2 = \lambda_2^2 = \frac{3k}{m} \Rightarrow \left(\frac{U_2}{U_3} \right)_{II} = -1$$

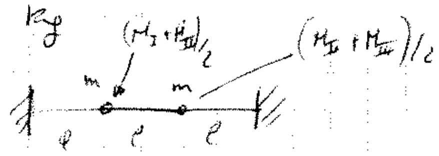
controfase

i) verificare le proprietà di K- e M-ortogonalità

ii) valutare effetto della massa dell'albero (LUMPED) (CONSISTENT)

$m = 15$ kg $\qquad \rho = 7800 \frac{kg}{m^3} \qquad \rho A \ell \simeq 0{,}74$ kg

$M_a = 3 \rho A \ell = 2{,}2$ kg



$$[M] = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \Rightarrow [M] = \begin{pmatrix} m + \rho A \ell & 0 \\ 0 & m + \rho A \ell \end{pmatrix}$$

i)

$$\left\{ X^{(1)} \right\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \left\{ X^{(2)} \right\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

M-ortogonalità

$$\left\{ X^{1} \right\}^T [M] \left\{ X^{(2)} \right\} = m \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 0$$

K - ortogonalità

$$\{x^{(1)}\}^T [K] \{x^{(1)}\} = K\,[1 \;\; +1] \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 0$$

$$\bar{K} = [\Psi]^T [K] [\Psi] \qquad [\Psi] = \left[\{x^{(1)}\} \; \{x^{(2)}\} \right] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\bar{K}_{ij} = \{x^{(i)}\}^T [K] \{x^{(j)}\} \qquad \bar{K}_{11} = 2K \qquad \bar{K}_{22} = 6K$$

$$\bar{K}_{12} = \bar{K}_{21} = 0 \qquad [\bar{K}] = \begin{bmatrix} 2K & 0 \\ 0 & 6K \end{bmatrix}$$

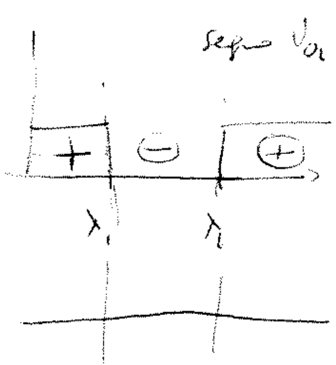
$$[\bar{M}] = [\Psi]^T [M] [\Psi] \qquad \bar{M}_{11} = 2m \qquad \bar{M}_{22} = 2m$$

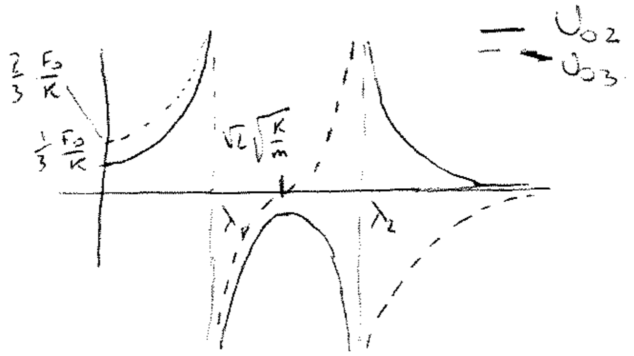
$$[\bar{M}] = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$$

Se viene applicata forzante in nodo 3 $(F_0 \cos(\lambda t))$

$$\begin{bmatrix} 2K - m\lambda^2 & -K \\ -K & 2K - m\lambda^2 \end{bmatrix} \begin{Bmatrix} U_{02} \\ U_{03} \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_0 \end{Bmatrix}$$

$$U_{02} = \frac{F_0}{\frac{m^2}{K}(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} \qquad U_{03} = \left(2 - \frac{m}{K}\lambda^2\right) U_{02} = \frac{F_0 \, (2\lambda_1^2 - \lambda^2)}{\lambda_1^2 \frac{m^2}{K} (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}$$



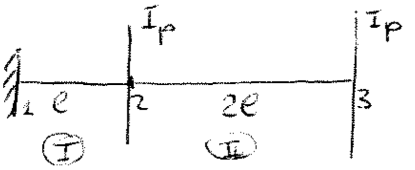


$$[\bar{M}] \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [\bar{K}] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [\Psi] \begin{Bmatrix} 0 \\ F_0 \end{Bmatrix} \cos\lambda t = \begin{Bmatrix} F_0 \\ -F_0 \end{Bmatrix} \cos\lambda t$$

$$Q_{0,i} = \frac{F_{0,i} / \bar{K}_{ii}}{1 - \left(\frac{\lambda}{\lambda_i}\right)^2} \qquad Q_{0,1} = \frac{F_0 / 2K}{1 - \lambda^2 \frac{m}{K}} \qquad Q_{0,2} = \frac{-F_0/2K}{1 - \frac{\lambda^2 m}{3K}}$$

$$\begin{Bmatrix} u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} Q_{0,2} \\ Q_{0,3} \end{Bmatrix} \cos \lambda t$$

Esercizio



$$M = \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix} = I_p \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = K \,\ell / G J_p \Rightarrow K = G \frac{J_p}{\ell} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

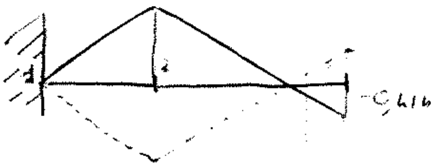
$$\left| [K] - \lambda^2 [M] \right| = 0 \qquad \lambda^4 - 4 \frac{t}{I_p} \lambda^2 + 2 \left(\frac{t}{I_p} \right)^2 = 0 \qquad \lambda^2 = (2 \pm \sqrt{2}) \frac{t}{I_p}$$

$$\left([K] - \lambda^2 [M] \right) \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\lambda^2 = \lambda_1^2 \quad \frac{\phi_2}{\phi_3} = 0{,}414$: $\phi_3 = 1$, $\phi_2 = 0{,}414$

$\lambda^2 = \lambda_2^2 \quad \frac{\phi_1}{\phi_3} = -2{,}415$: $\phi_2 = 1$, $\phi_3 = -0{,}414$

$\lambda^2 = \lambda_1^2$

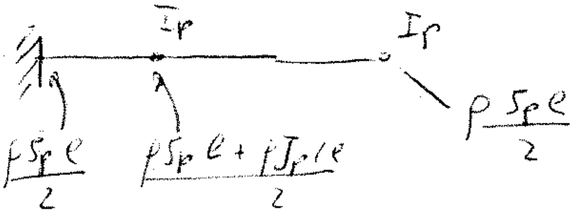


i) K- e M-ortogonalità

ii) effetto inerzia torsionale dell'albero

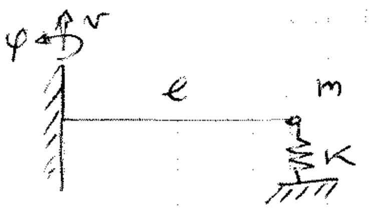
ii) $I_p = 4 \cdot 10^{-2}$ Kg m²

albero $\rho J_p \ell = \rho \frac{\pi d^4}{32} \ell \approx 810 \cdot 10^{-8}$ Kg m² $<< I_p$



In questo caso non è significativo valutare ~~inerzia torsionale~~ andare oltre con i valori

Esercizio



$$\lambda^2 = \frac{K}{m}$$

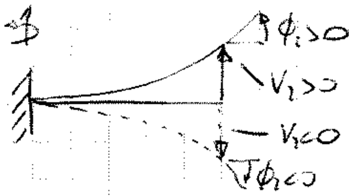
Se non c'è vincolo elastico $K = \frac{3EJ}{\ell^3}$

Se c'è vincolo elastico $K = \frac{3EJ}{\ell^3} + k$

$$[M] = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} v_2 \\ \varphi_2 \end{matrix} \qquad [K] = \begin{bmatrix} 12 + \frac{k\ell^3}{EJ} & -6\ell \\ -6\ell & 4\ell^2 \end{bmatrix} \frac{EJ}{\ell^3}$$

$$\left| [K] - \lambda^2 [M] \right| = 0 \qquad m\lambda^2 - \left(\frac{3EJ}{\ell^3} + k \right) = 0 \qquad \lambda^2 = \frac{\frac{3EJ}{\ell^3} + k}{m}$$

$$\left([K] - \lambda^2 [M] \right) \begin{Bmatrix} V_2 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \frac{\varphi_2}{V_2} = \frac{3}{2\ell}$$



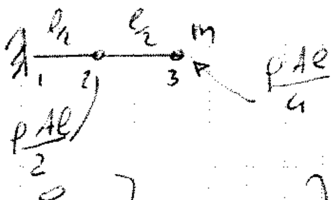
i) Massa albero < lumped / consistente

lumped

$$M = \begin{pmatrix} m + \frac{\rho A \ell}{2} & 0 \\ 0 & 0 \end{pmatrix} \qquad \lambda_{1,L} = 10,15 \ \frac{rad}{s}$$

K non cambia $\qquad \lambda_1 = 11,36 \ \frac{rad}{s}$

2 elementi



$$[M] = \begin{bmatrix} \frac{\rho A \ell}{2} & 0 \\ 0 & m + \frac{\rho A \ell}{4} \end{bmatrix} \qquad \lambda_{1,L} \simeq 10,6 \ \frac{rad}{s} \qquad \lambda_{1,C} \simeq 11,8 \ rad/s$$

I valori esatti $\lambda_{1,TH} = 10,74 \ \frac{rad}{s}$ $\qquad \lambda_{2,TH} = 175,6 \ \frac{rad}{s}$

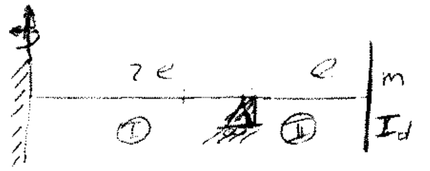
Approccio consistente

K rimane invariata

$$[M] = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} + \underbrace{\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}}_{\text{effetto massa dell'albero}}$$

$$\left|[K] - \lambda^2 [M]\right| = 0 \quad \rightarrow \quad \lambda_{1,c} \cong 10{,}75 \text{ rad/s} \qquad \lambda_{2,c} \cong 169 \text{ rad/s}$$

11/01/2012



$$\begin{array}{cc} [K] = \dfrac{EJ}{l^3}\begin{bmatrix} 6l^2 & -6l & 2l^2 \\ -6l & 11 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix} & \begin{array}{c} \varphi_2 \\ v_3 \\ \varphi_3 \end{array} \end{array}$$

$$[M] = \begin{bmatrix} 0 & & \emptyset \\ & m & \\ \emptyset & & I_d \end{bmatrix}$$

$$\left|K - \lambda^2 [m]\right| = 0 \Rightarrow \lambda^4 - \frac{2EJ}{m l I_d}\left(\frac{5}{3}m + \frac{3I_d}{l^2}\right)\lambda^2 + \frac{4\,(EJ)^2}{m l^4 I_d} = 0$$

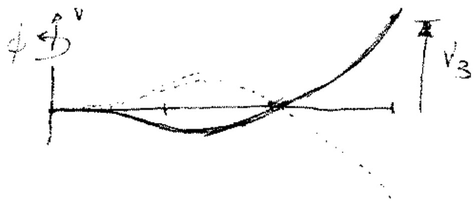
$\lambda_1 \approx 13$ rad/s

$\lambda_2 \approx 131$ rad/s

$$([K] - \lambda^2 [M]) \begin{Bmatrix} \varphi_2 \\ v_3 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Fissiamo arbitrariamente a 1 un autovalore (es: φ_3) e si ricavano gli altri due

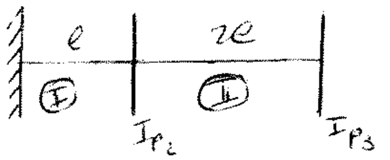
Se $\lambda^2 = \lambda_1^2$
$$\begin{Bmatrix} \varphi_2 \\ v_3 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0{,}49 \\ 0{,}41 \\ 1 \end{Bmatrix}$$



Se $\lambda^2 = \lambda_2^2$
$$\begin{Bmatrix} \varphi_2 \\ v_3 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} -0{,}87 \\ -0{,}02 \\ 1 \end{Bmatrix}$$

Se devo considerare la massa dell'albero non cambia molto nello approccio LUMPED perché i nodi 1 e 2 sono collegati e si ha effetto della massa solo nel nodo 3.

Esercizio: risposta forzata su sistema torsionale



$$\frac{G J_p}{l} = t$$

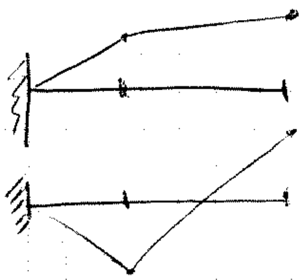
$$[K] = \begin{bmatrix} t_I + t_{II} & -t_{II} \\ -t_{II} & t_{II} \end{bmatrix} \qquad [M] = \begin{bmatrix} I_{p2} & 0 \\ 0 & I_{p3} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$|[K] - \lambda^2[M]| = 0 \qquad I_{p3} I_{p2} \lambda^4 - \left[(t_I + t_{II}) I_{p3} + t_{II} I_{p2} \right] \lambda^2 + t_I t_{II} = 0$$

$\lambda_1 \simeq 53$ rad/s $\qquad \lambda_2 \simeq 152$ rad/s

Se $\lambda^2 = \lambda_1^2 \qquad \theta_2/\theta_3 = 0,77$

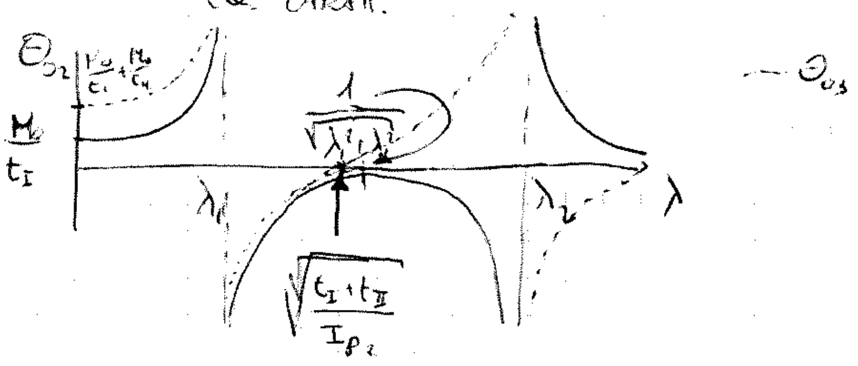
Se $\lambda^2 = \lambda_2^2 \qquad \theta_2/\theta_3 = -0,86$



$$M_t = M_0 \sin(\lambda t)$$

$$\left([K] - \lambda^2[M]\right) \begin{Bmatrix} \theta_{02} \\ \theta_{03} \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}$$

$$\theta_{02} = \frac{M_0 t_{II}}{\text{EQ. CARATT.}} \qquad \theta_{03} = \frac{\left[(t_I + t_{II}) - I_{p2}\lambda^2\right] M_0}{\text{eq. caratt.}}$$

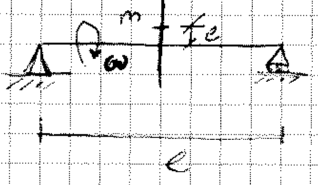


i) calcolo risp. forzata con analisi modale o solo con calcolo tradizionale

ii) libere solite ult richieste

18/01/2012

Esercizio: supponiamo di poter schematizzare l'albero come albero di Jeffcott



$m = 15\ kg$ $e = 10\ \mu m$

$l = 500\ mm$ $\omega = 10^4$ giri/minuto

Domande: 1) calcolare ω_{CR}

2) dimensionare l'albero in modo che $\frac{f}{e} = -1,2$ oppure $\frac{f}{e} = +1,2$

3) carico sui cuscinetti

4) σ_{max} sull'albero

$$\omega_{CR} = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{48EJ}{l^3}}{m}} =$$

$$J = \frac{\pi d^4}{64} =$$ non conosco ancora d, ma so che $\frac{f}{e} = -1,2$

$$\frac{f}{e} = \frac{\left(\frac{\omega}{\omega_{CR}}\right)^2}{1 - \left(\frac{\omega}{\omega_{CR}}\right)^2} = -1,2 \Rightarrow \left(\frac{\omega}{\omega_{CR}}\right)^2 = 6 \qquad \omega_{CR} = \sqrt{\frac{\omega^2}{6}} = \frac{\omega}{\sqrt{6}} = 4082\ \frac{giri}{min}$$

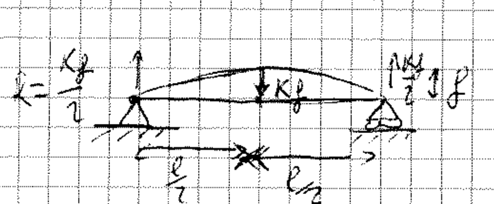
$$d = \sqrt[4]{\frac{4ml^3}{3\pi E}\omega_{CR}^2} = 29\ mm$$

Se voglio $\frac{f}{e} = 1,2$ (regime subcritico)

$$(\omega/\omega_{CR})^2 = \frac{6}{11} \qquad \omega_{CR} = \sqrt{\frac{11}{6}\omega^2} \cong 13500\ giri/minuto$$

$$d = \sqrt[4]{\frac{4ml^3}{3\pi E}\omega_{CR}} = 53\ mm$$

Gli ultimi 2 punti sono problemi di statica

3) 

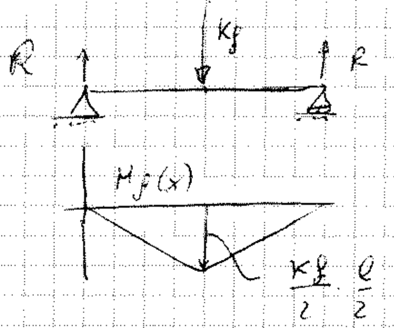
tiene che m mezzeria sente spostamento pari a f

$$K = \frac{48EJ}{l^3} \qquad [K] = N/m$$

$$R = \frac{Kf}{2} = \qquad R\Big|_{\frac{f}{e}=-1,2} = \frac{1}{2}\frac{48EJ}{l^3}|f| \qquad f = |1,2|\cdot e$$

$$R = K\omega_{CR}^2 m\,|f| = \begin{cases} \frac{f}{e} = -1{,}2 & = 33\ N \\ \frac{f}{e} = +1{,}2 & = \end{cases} \qquad \frac{R_{sup}}{R_{sub}} = \left(\frac{53}{29}\right)^4$$

4) $\sigma_{max} = ?$

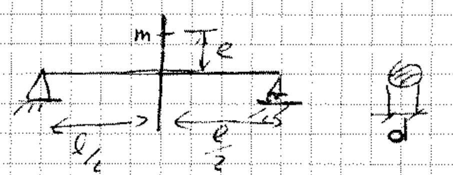


$$\sigma_{max} = \frac{M_f}{W_f} \qquad W_f = \frac{J}{\frac{d}{2}} = \frac{\pi d^3}{32}$$

$$\sigma_{max}\Big|_{\frac{f}{e}=-1,2} = \frac{F\ell 8}{\pi d^3} = 1{,}7\ MPa$$

$$\sigma_{max}\Big|_{\frac{f}{e}=1,2} \Rightarrow \frac{\sigma_{max}\big|_{sub}}{\sigma_{max}\big|_{sup}} = \left(\frac{53}{29}\right)$$

Esercizio



$d = 15\ mm \qquad \ell = 600\ mm$

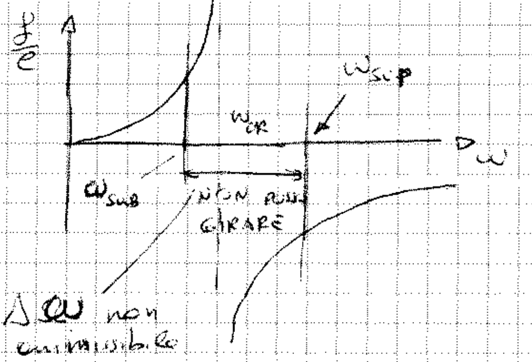
$e = 0{,}33\ mm \qquad m = 9\ kg$

1) Determinare ω_{CR}, in modo che σ_{max} nell'albero sia al massimo 150 MPa $\quad E = 200000\ MPa$

2) determinare intervallo di velocità

1) $$\omega_{CR} = \sqrt{\frac{K}{m}} = \sqrt{\frac{48EJ}{\ell^3 m}} = \left(141\ rad/s \cdot \frac{60}{2\pi}\right) rpm$$

2)



Distinguiamo comportamento sub-critico da supercritico.

$$SUB \quad \frac{f}{e} = \frac{\left(\frac{\omega}{\omega_{CR}}\right)^2}{1-\left(\frac{\omega}{\omega_{CR}}\right)^2}$$

$$\sigma_{max} = \frac{K\ell}{4W_f} f \le \sigma_{amm}$$

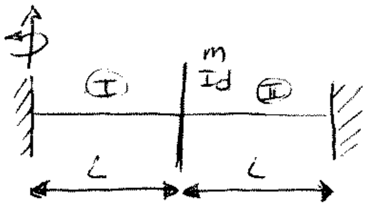
$$\frac{K\ell\, e\, \frac{(\omega/\omega_{CR})^2}{1-(\omega/\omega_{CR})^2}}{4W_f} \le \sigma_{amm} \Rightarrow \left(\frac{\omega}{\omega_{CR}}\right)^2 \le \frac{\sigma_{amm}}{\sigma_{amm} + \frac{K\ell e}{4W_f}}$$

$$\omega_{SUB} = \omega_{CR}\sqrt{\frac{\sigma_{amm}}{\sigma_{amm} + \frac{K\ell e}{4W_f}}}$$

$$SUPER \quad \frac{f}{e} = \frac{(\omega/\omega_{CR})^2}{(\omega/\omega_{CR})^2 - 1} \qquad \frac{K\ell e\, \frac{(\omega/\omega_{CR})^2}{(\omega/\omega_{CR})^2-1}}{4W_f} \le \sigma_{amm} \Rightarrow \left(\frac{\omega}{\omega_{CR}}\right)^2 \ge \frac{\sigma_{amm}}{\sigma_{amm} - \frac{K\ell e}{4W_f}}$$

$$\omega_{sup} = \omega_{CR}\sqrt{\frac{\sigma_{amm}}{\sigma_{amm} - \frac{K\ell e}{4W_f}}}$$

Esercizio



$v_1 = \varphi_1 = v_3 = \varphi_3 = 0$

$$[K] = \frac{EJ}{\ell^3}\begin{bmatrix} 24 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \qquad [M] = \begin{bmatrix} m & 0 \\ 0 & I_d \end{bmatrix}$$

$$\left| [K] - \lambda^2 [M] \right| = 0 \qquad \begin{Vmatrix} 24K - \lambda_1^2 m & 0 \\ 0 & 8\ell^2 K - \lambda_2^2 I_d \end{Vmatrix} = 0$$

$K = \frac{EJ}{\ell^3}$

$$\lambda_1^2 = 24\frac{K}{m}$$

$$\lambda_2^2 = 8\ell^2 \frac{k}{I_d}$$

$$\begin{pmatrix} 24k - \lambda^2 m & 0 \\ 0 & 8k\ell^2 - \lambda^2 I_d \end{pmatrix} \begin{Bmatrix} V_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Se $\lambda^2 = \lambda_1^2$

$$\begin{pmatrix} 0 & 0 \\ 0 & 8k\ell^2 - \lambda_1^2 I_d \end{pmatrix} \begin{Bmatrix} V_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

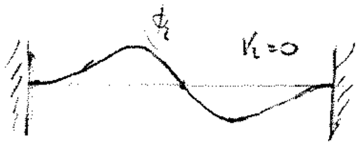
I equazione non dà condizioni

$(8k\ell^2 - \lambda_1^2 I_d)\phi_2 = 0 \rightarrow \phi_2 = 0$

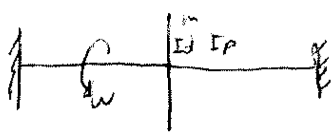
$\lambda^2 = \lambda_2^2$

$$\begin{pmatrix} 24k - \lambda_2^2 m & 0 \\ 0 & 0 \end{pmatrix} \begin{Bmatrix} V_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad V_2 = 0$$

ϕ_2 non dà condizioni



albero in rotazione

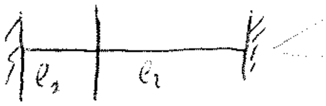


$I_p = 2 I_d$ (disco sottile) $\qquad [K]$ è sempre lo stesso

$$[M] = \begin{bmatrix} m & 0 \\ 0 & -(I_p - I_d) \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & -I_d \end{bmatrix} \qquad \left| [K] - \omega^2 [M] \right| = 0$$

$$\begin{vmatrix} 2hk-\omega^2 m & 0 \\ 0 & \underbrace{8k\ell^2+J_d\omega^2}_{\cancel{\omega^2 0}} \end{vmatrix} = 0 \qquad \omega^2 = \frac{2hK}{m} \qquad K = \frac{ES}{\ell^3}$$

Esercizio a casa



albero vibrante

albero rotante