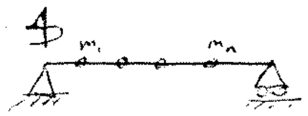
## Elimento trave

$$[M] = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & & \text{simm.} & \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

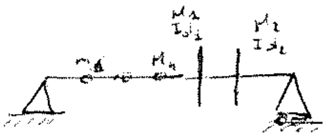
Nel caso di parametri concentrati

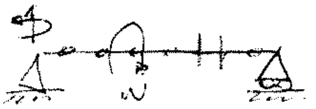
$$[M] = \frac{\rho A \ell}{2} \begin{bmatrix} 1 & & \emptyset \\ & 0 & \\ & & 1 \\ 0 & & & 0 \end{bmatrix}$$

## DINAMICA dei ROTORI



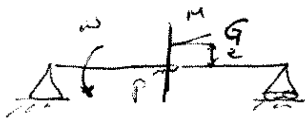

Freq. proprie $\lambda_r^2$ - Modi di vibrare $\{X^{(r)}\}$



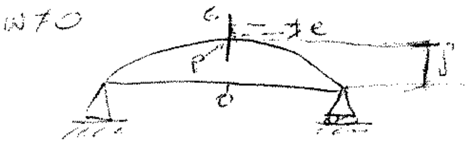

è il problema di dinamica dei rotori.

### Rotore di Jeffcott



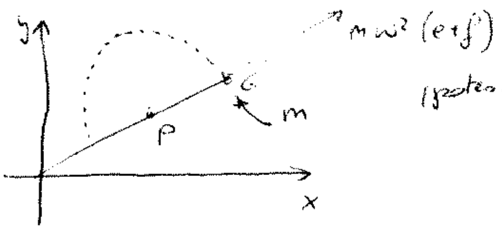

e eccentricità della massa

soluzione elementare



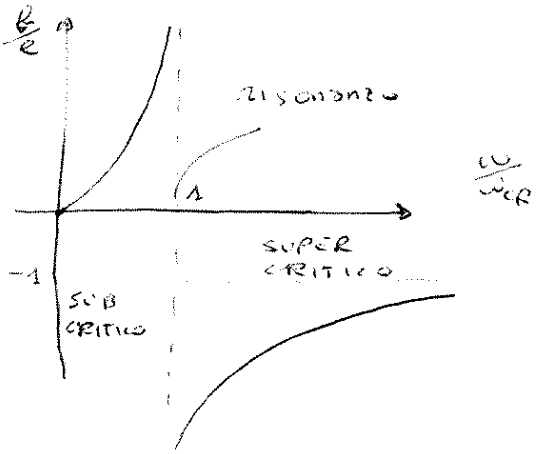

$$m\omega^2 (f + e) = K f$$

$$K = \frac{48 EJ}{\ell^3} \qquad J = \frac{\pi d^4}{64}$$




Ipotesi: rotazione rigida della deformata dell'albero

$$f = \frac{m\omega^2 e}{K - m\omega^2} = e\,\frac{\left(\frac{\omega}{\omega_{CR}}\right)^2}{1-\left(\frac{\omega}{\omega_{CR}}\right)^2} \qquad \omega_{CR} = \sqrt{\frac{K}{m}} \to \lambda_n \cdot \sqrt{\frac{K}{m}}$$

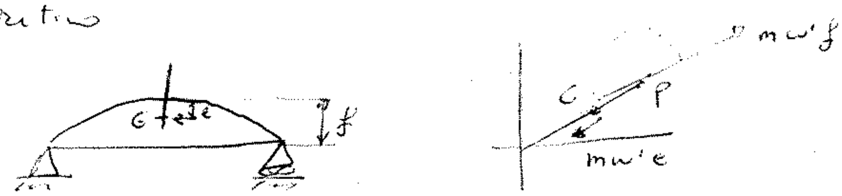



Se $\omega \to \infty$ AUTOCENTRAMENTO

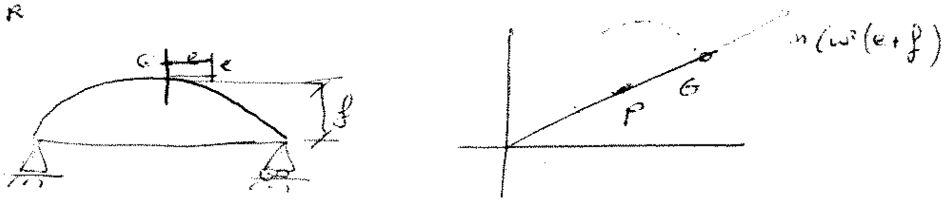
Se $\omega > \omega_{CR}$

$$\omega^2 (f + e) = \omega^2_{CR} f$$
$$\omega^2 f = \omega^2_{CR} (f - e)$$

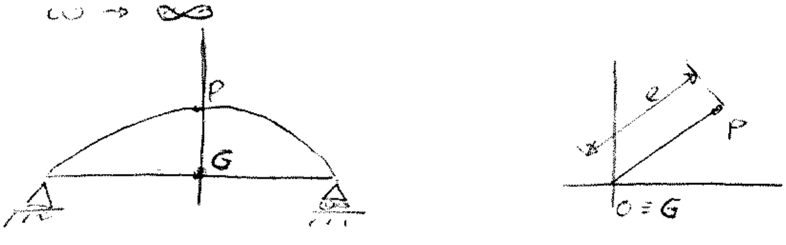
Supercritico




Se $\omega < \omega_{CR}$

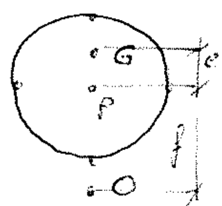



Se $\omega \to \infty$



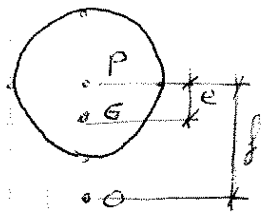

$\omega = 0$


$0 < \omega < \omega_{CR}$

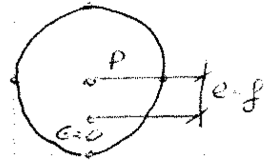


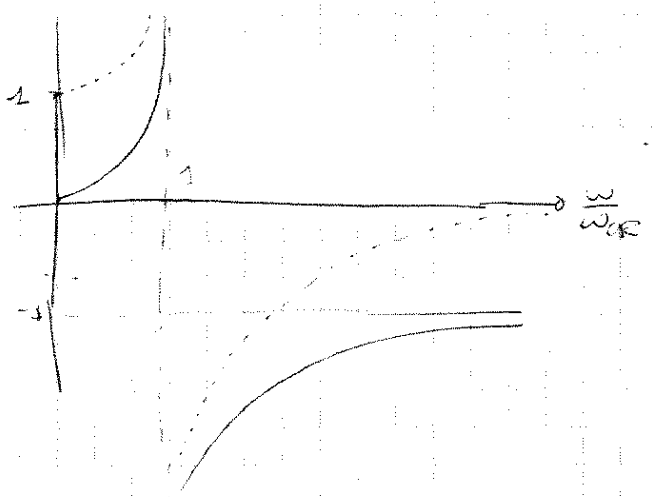

Regime subcritico

$\omega > \omega_{CR}$




$\omega \to \infty$




— Jeffcott

..... O.A. curva dell' oscillatore armonico

Jeffcott $\omega_{CR} = \sqrt{\frac{K}{m}}$ cioè la velocità critica caratterizza

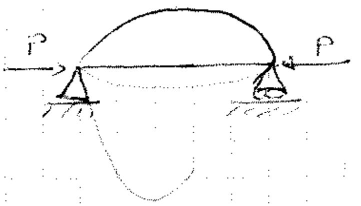
la risposta forzata $(e \neq 0)$

Se consideriamo la risposta libera $(e = 0)$ abbiamo che

$m\omega^2 f = Kf \qquad \omega^2 f = \omega_{CR}^2 f$

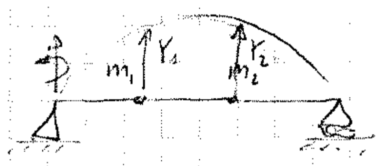
se $\omega \neq \omega_{CR}$ $f = 0$

se $\omega = \omega_{CR}$ $\forall f$ va bene. Cioè per un sistema con disco perfettamente centrato l'albero è nelle condizione di equilibrio indifferente.




$P_{CR} = EJ\left(\frac{\pi}{l}\right)^2$

2gdl




$m_1 \omega^2 Y_1 = K_{11} Y_1 + K_{12} Y_2$

$m_2 \omega^2 Y_2 = K_{21} Y_1 + K_{12} Y_2$

$$\left[\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} - \omega^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\right]\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left([K]-\omega^2[M]\right)\{Y\} = \{0\}$$

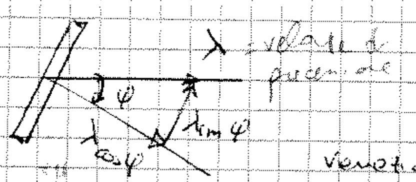
ROTORI - VIBRAZIONI

$$\left.\begin{array}{l} \omega_K^2 = \lambda_K^2 \\ \left|[K]-\omega^2[M]\right| = 0 \Rightarrow \left|[K]-\lambda^2[M]\right| = 0 \end{array}\right\} \text{identità numerica}$$

$$\{Y^{(K)}\} = \{X^{(K)}\} \quad \text{identità numerica}$$

## Momento giroscopico sull'albero

linea elastica




$dT = dL$ teorema dell'energia cinetica

$dT$: variazione energia cinetica (del disco)

$dL$: lavoro compiuto (dal disco)

$$T = \frac{1}{2} I \Omega^2$$

$$T = \frac{1}{2}\left[I_p\left(\mu + \lambda\cos\varphi\right)^2\right] + \frac{1}{2} I_d\left(\lambda \sin\varphi\right)^2$$

$$dT = I_p\left(\mu + \lambda\cos\varphi\right)\left(-\lambda\sin\varphi\right)d\varphi + I_d\left(\lambda\sin\varphi\right)\lambda\cos\varphi\, d\varphi =$$

$$= -I_p \lambda\omega\,\varphi\, d\varphi + I_d \lambda^2 \varphi\, d\varphi = -\left(I_p\lambda\omega - I_d\lambda^2\right)\varphi\, d\varphi$$

$\varphi \ll 1$ $\quad \mu + \lambda = \omega$

$\lambda = \omega$ (ipotesi rotazione rigida della deformata)

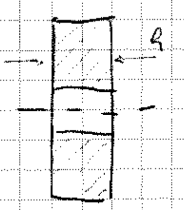
$$dT = -\left(I_p - I_d\right)\omega^2 \varphi\, d\varphi$$

$$dL = M_g\, d\varphi$$

$M_g$ = M giroscopico sul disco

$$M_{g\,\text{sul disco}} = -\left(I_p - I_d\right)\omega^2\varphi$$

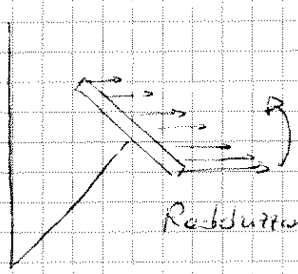
$$M_{g\,\text{sull'albero}} = \left(I_p - I_d\right)\omega^2\varphi$$



$$I_p = m\,\frac{r^2+R^2}{2}$$

$$I_d = m\,\frac{r^2+R^2+\frac{h^2}{3}}{4}$$

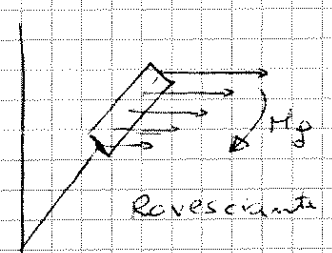
$I_p > I_d \qquad I_p - I_d > 0$




Disco sottile $\quad I_p - I_d = I_d > 0$

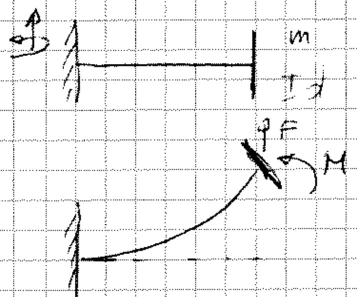
Raddrizzante

Se $I_p < I_d \Leftrightarrow I_p - I_d < 0$



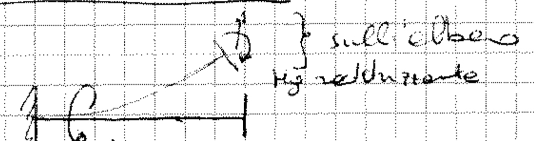

Rovesciante

## Albero vibrante




$$M = \begin{pmatrix} m & 0 \\ 0 & I_d \end{pmatrix}$$

## Albero rotante

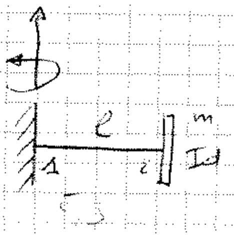



$$[M] = \begin{pmatrix} m & 0 \\ 0 & -(I_p - I_d) \end{pmatrix} = \underbrace{\begin{pmatrix} m & 0 \\ 0 & -I_d \end{pmatrix}}_{\text{se disco sottile}}$$

INerzie: masse concentrate + masse dischi

freq proprie: 1 freq propria per ogni inerzia → (n+2m) freq totali

velocità critiche: 1 ~~freq~~ vel critica per ogni massa + 1 per ogni disco → (n+m) velocità critiche

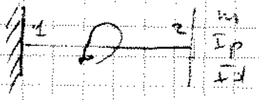



$$[M] = \begin{pmatrix} m & 0 \\ 0 & I_d \end{pmatrix} \qquad [K] = \frac{EJ}{l^3}\begin{pmatrix} 12 & -6l \\ -6l & 4l^2 \end{pmatrix}$$

$$\left|[K] - \lambda^2 [M]\right| = 0 \rightarrow \lambda^4 - \frac{4EJ}{m l I_d}\left(m + \frac{3I_d}{l^2}\right)\lambda^2 + \frac{12(EJ)^2}{m l^4 I_d} = 0$$

$$\lambda^2 \left(\frac{2EJ}{l I_d} + \frac{6EJ}{m l^3}\right) \pm \sqrt{\left(\;\right)^2 - \frac{12(EJ)^2}{m l^4 I_d}} \qquad 0 < \lambda_1^2 < \lambda_2^2$$

Albero rotante




$[K] =$ come prima

$$[M] = \begin{pmatrix} m & 0 \\ 0 & -(I_p - I_d) \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & -A \end{pmatrix} \qquad A > 0$$

$$\left|[K] - \omega^2 [M]\right| = 0 \qquad \omega^4 + \frac{4EJ}{m l A}\left(m - \frac{3A}{l^2}\right)\omega^2 - \frac{12(EJ)^2}{m l^4 A} = 0$$

$$I_d \rightarrow -A = -(I_p - I_d)$$

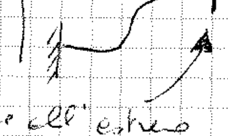
$$\omega^2 = -\left(\frac{2EJ}{l A} - \frac{6EJ}{m l^3}\right) \pm \sqrt{\left(\;\right)^2 + \frac{12(EJ)^2}{m l^4 A}}$$

$$\Omega = \omega_{CR}\sqrt{\frac{m l^3}{EJ}}, \quad \alpha = \frac{A}{m l^2} \quad \text{in} \quad \omega_t^2 = \omega_{CR}^2$$

$$\Omega^2 = -\left(\frac{2}{\alpha} - 6\right) + \sqrt{\left(\frac{2}{\alpha} - 6\right)^2 + \frac{12}{\alpha^2}}$$


$\alpha \rightarrow \infty \quad \Omega^2 \rightarrow 12$

$$\omega_{CR} \Rightarrow \frac{12 EJ}{m l^3} \quad \left\{ \varphi_2 = 0 \right.$$



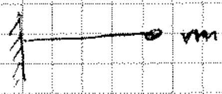
rotazione all'estremo in cui è collocato il disco è impedita

Se $\alpha \to 0$ non abbiamo effetto giroscopico (c'è solo più massa)

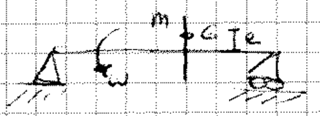
$\downarrow$

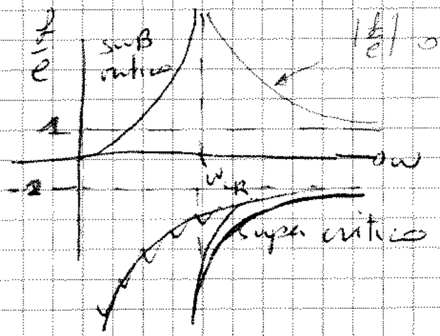
$A \to 0$ $\qquad \frac{4EJ}{m\ell}\, m\omega^2 - \frac{12(EJ)^2}{m\ell^4} = 0 \qquad \omega^2 = \frac{3EJ}{\ell^3 m}$

$\downarrow$

[K] di 


albero con sola massa concentrata




$\left|\frac{f}{e}\right|$ o $\frac{f}{e}$ dell'approssimazione supercritica

Sub critico $\quad m\omega^2(e+f) = Kf \;\longrightarrow\; \frac{f}{e} = \frac{\left(\frac{\omega}{\omega_{CR}}\right)^2}{1-\left(\frac{\omega}{\omega_{CR}}\right)^2}$

Supercritico $\quad m\omega^2 f = Kf + m\omega^2 e \;\longrightarrow\; \frac{f}{e} = \frac{(\omega/\omega_{CR})^2}{\left(\frac{\omega}{\omega_{CR}}\right)^2 - 1}$