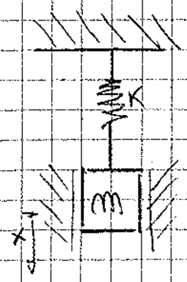
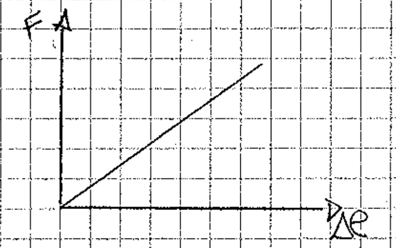
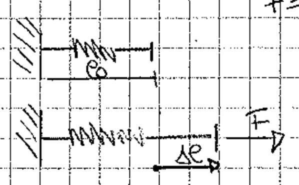
LE VIBRAZIONI

VIBRAZIONI LIBERE

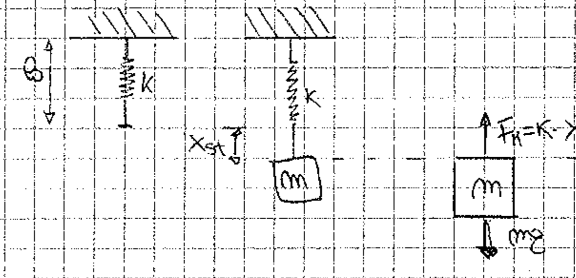


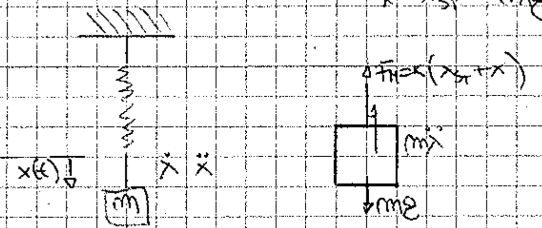


$F = K \cdot \Delta e$



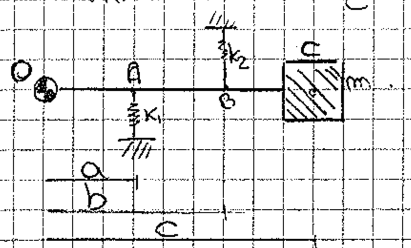
$K \cdot x_{ST} = mg$

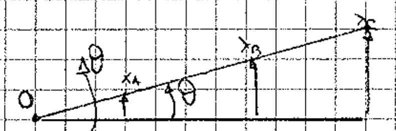




$K \cdot x_{ST} = mg$

$m\ddot{x} + Kx + \underbrace{Kx_{ST} - mg}_{=0} = 0 \qquad \underline{m\ddot{x} + Kx = 0} \quad \to x(t) = \_\_$



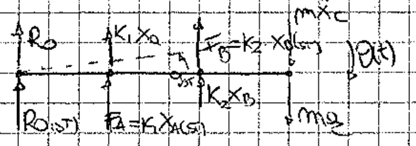


$\ddot{x}_A = a \cdot tg\,\ddot{\theta} \cong a\ddot{\theta}$

$\ddot{x}_B \cong b\ddot{\theta}$

$\ddot{x}_C \cong c\ddot{\theta}$

È UN SISTEMA A DUE GRADI DI LIBERTÀ



$F_A = K_1 \cdot x_{A(ST)} = K_1 \cdot a \cdot \theta_{ST}$

$F_B = K_2 \cdot x_{B(ST)} = K_2 \cdot b \cdot \theta_{ST}$

$K_1 \cdot a^2 \cdot \theta_{ST} + K_2 \cdot b^2 \theta_{ST} = mg \cdot c$

Le quattro forze rosse sono variabili nel tempo $\to$ DINAMICHE

$\underbrace{K_1 \cdot a^2 \theta_{ST} + K_2 b^2 \theta_{ST} - mg \cdot c}_{=0} + \underbrace{K_1 x_A \cdot a + K_2 \cdot x_B \cdot b + m\ddot{x}_C \cdot c = 0}_{=0}$

$\underline{K_1 a^2 \theta + K_2 b^2 \theta + m c^2 \ddot{\theta} = 0}$

EQUAZIONE DEL MOTO DEL SISTEMA

$\Downarrow$

$\theta(t) = \_\_$

$mc^2 \ddot{\theta} + (K_1 a^2 + K_2 b^2)\theta = 0$

$\ddot{\theta} + \Big(\underbrace{\frac{K_1 a^2 + K_2 b^2}{mc^2}}_{\omega_n^2}\Big)\theta = 0$

$\ddot{x} + \omega_n^2 x = 0 \quad \to \quad x(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t}$

$x(t) = e^{\lambda t} \qquad \ddot{x} = \lambda^2 e^{\lambda t}$

$\lambda^2 \cancel{e^{\lambda t}} + \omega_n^2 \cancel{e^{\lambda t}} = 0$

$\lambda_{1,2} = \pm i\omega_n$

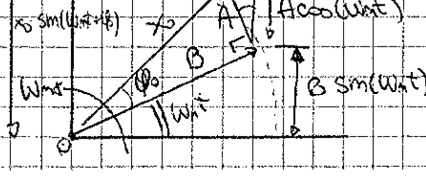
$\underline{x = a \cdot e^{i\omega_n t} + b e^{-i\omega_n t}}$

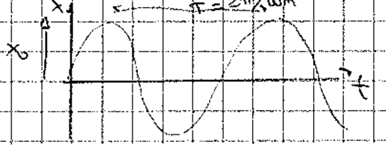
$e^{\pm i\alpha} = \cos\alpha \pm i \sin\alpha$

$x = A \cdot \cos(\omega_n t) + B \sin(\omega_n t) =$

$= x_0 \sin(\omega_n t + \varphi_0)$

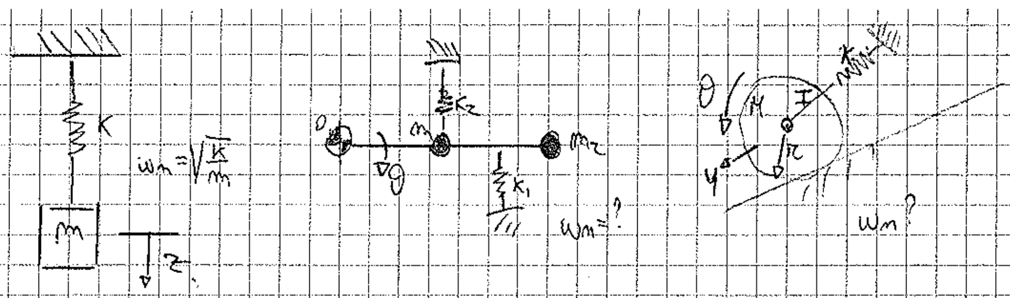
PULSAZIONE NATURALE DEL SISTEMA





$\begin{cases} f(t) = f(t+T) \\ \sin\alpha = \sin(\alpha + 2\pi) \\ x_0 \sin(\omega_n t + \varphi_0 + 2\pi) = x_0 \sin(\omega_n t + \varphi_0 + \omega_n T) \end{cases}$

$\omega_n T = 2\pi \qquad \omega_n = \frac{2\pi}{T}$



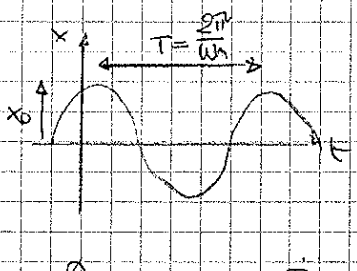
Questi tre esempi si possono ricondurre a questa equazione del moto

$$\boxed{\ddot{x} + \omega_n^2 x = 0}$$

nel parametro ω_n PULSAZIONE NATURALE DEL SISTEMA sono comprese tutte le caratteristiche del sistema

$$x = x_0 \sin(\omega_n t + \varphi_0)$$

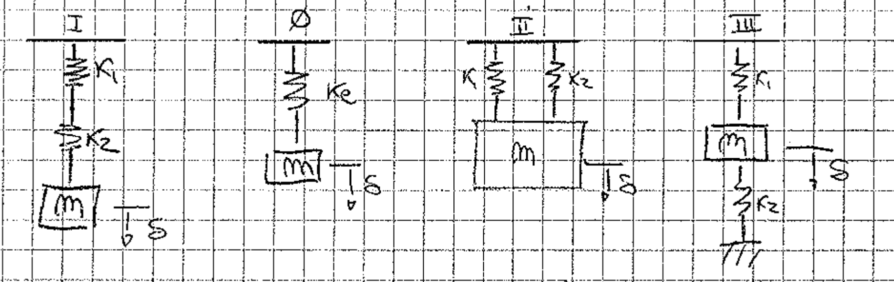
LEGGE DEL MOTO: è sempre una sinusoide



$T = \frac{1}{f_n}$ $\quad \omega_n = 2\pi f_n$ $\quad [\omega_n] = \frac{rad}{s}$

$[f_n] = \frac{1}{s} = Hz$

$[T] = s$



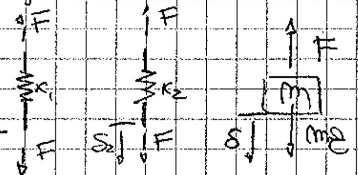
Ricavare $k_e = f(k_1, k_2)$

∅

$F = k_e \cdot \delta = mg$

$\delta = \frac{mg}{k_e}$

I SERIE



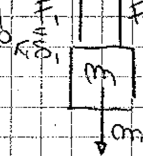
$$\boxed{k_e = \frac{k_1 k_2}{k_1 + k_2}}$$

$mg = k_1 \cdot \delta_1 \qquad \delta_1 = \frac{mg}{k_1}$

$F = mg = k_2 \cdot \delta_2 \qquad \delta_2 = \frac{mg}{k_2}$

$\delta = \delta_1 + \delta_2 = mg \frac{k_1 + k_2}{k_1 \cdot k_2}$

II PARALLELO



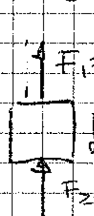
$F_1 = k_1 \delta_1 \qquad F_2 = k_2 \cdot \delta_2$

$\delta = \delta_1 = \delta_2$

$mg = F_1 + F_2 = k_1 \cdot \delta_1 + k_2 \cdot \delta_2 = (k_1 + k_2)\,\delta$

$mg = k_e \cdot \delta \qquad \boxed{k_e = k_1 + k_2}$

III PARALLELO

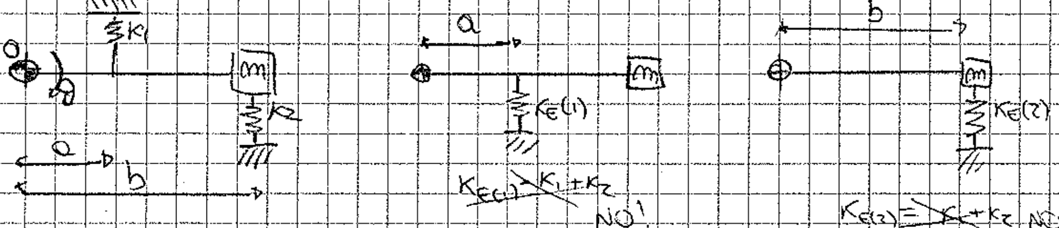


$F_1 = k_1 \cdot \delta_1 = k_1 \cdot \delta$

$mg = F_1 + F_2 = (k_1 + k_2)\,\delta$

$F_2 = k_2 \cdot \delta_2 = k_2 \cdot \delta$

$k_e = k_1 + k_2$



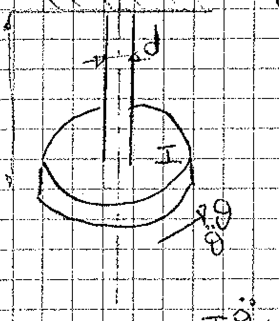
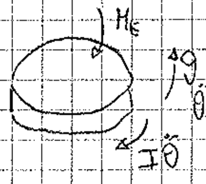
$\cancel{k_{E(1)} = k_1 + k_2}$ NO!

$\cancel{k_{E(2)} = k_1 + k_2}$ NO!

$k_{E(1)} = f_1(k_1, k_2, a, b)$

$k_{E(2)} = f_2(k_1, k_2, a, b)$

OSCILLAZIONI TORSIONALI

$$I\ddot{\theta} + M_E = 0$$

$$M_E = \frac{GJ}{\ell}\theta$$

MOMENTO D'INERZIA POLARE

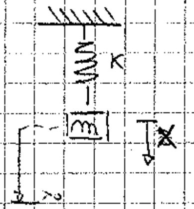
$$J = \frac{\pi}{32}d^4$$

G MODULO DI ELASTICITÀ TANGENZIALE

$$I\ddot{\theta} + \frac{GJ}{\ell}\theta = 0 \qquad I\ddot{\theta} + k\theta = 0 \qquad \ddot{\theta} + \omega_n^2\theta = 0$$

RIGIDEZZA TORSIONALE

$$\theta = \theta_0 \sin(\omega_n t + \varphi_0)$$

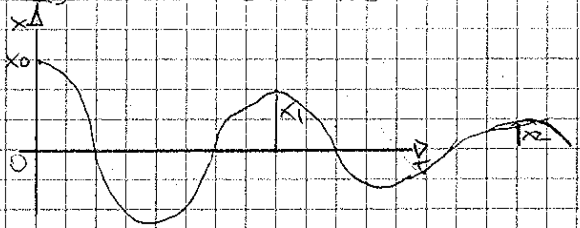


$$\ddot{x} + \omega_n^2 x = 0$$

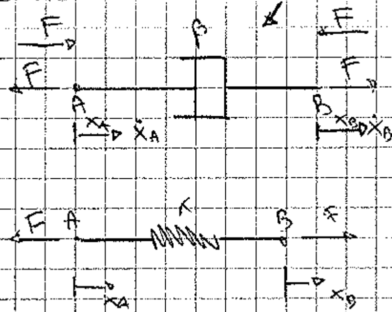
$$x = x_0 \sin(\omega_n t + \varphi_0)$$

$$x = a \cdot e^{i\omega_n t} + b e^{-i\omega_n t}$$

Lo lascio andare all'istante $t=0$ dalla posizione x_0



SMORZATORE VISCOSO



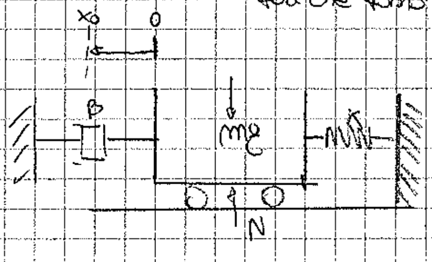
$$F = \beta(\dot{x}_B - \dot{x}_A)$$

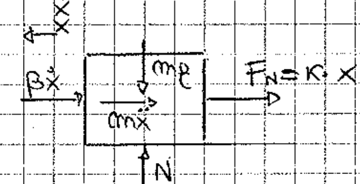
$$F = \beta(\dot{x}_A - \dot{x}_B)$$

$$F = K \cdot (x_B - x_A)$$

Lo smorzatore viscoso si può realizzare con un cilindro pistone pieno d'olio

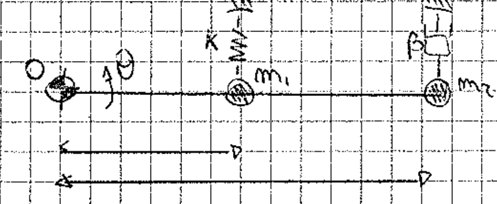
fori che fanno passare l'olio incomprimibile

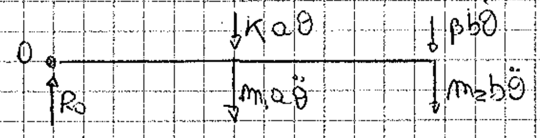




$$m\ddot{x} + \beta\dot{x} + kx = 0$$

$$\ddot{x} + \underbrace{\frac{\beta}{m}}_{2\zeta\omega_n}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_n^2}x = 0$$





$I_O\ddot{\theta}$

$$m_1 a^2\ddot{\theta} + m_2 b^2\ddot{\theta} + \beta b^2\dot{\theta} + k a^2\theta = 0$$

$$(m_1 a^2 + m_2 b^2)\ddot{\theta} + \beta b^2\dot{\theta} + k a^2\theta = 0$$

$$\ddot{\theta} + \underbrace{\frac{\beta b^2}{m_1 a^2 + m_2 b^2}}_{2\zeta\omega_n}\dot{\theta} + \underbrace{\frac{k a^2}{m_1 a^2 + m_2 b^2}}_{\omega_n^2} = 0$$

ζ : FATTORE DI SMORZAMENTO

$\ddot{x} + 2\zeta\omega_m\dot{x} + \omega_m^2 x = 0$ FORMA GENERALIZZATA

$x = a e^{\lambda_1 t} + b e^{\lambda_2 t}$

$x = e^{\lambda t}$ $\dot{x} = \lambda e^{\lambda t}$ $\ddot{x} = \lambda^2 e^{\lambda t}$

$\lambda^2 \cancel{e^{\lambda t}} + 2\zeta\omega_m\lambda \cancel{e^{\lambda t}} + \omega_m^2 \cancel{e^{\lambda t}} = 0$

$\lambda^2 + 2\zeta\omega_m\lambda + \omega_m^2 = 0$ $\lambda_{1,2} = -\zeta\omega_m \pm \sqrt{\zeta^2\omega_m^2 - \omega_m^2} = \omega_m(-\zeta \pm \sqrt{\zeta^2 - 1})$

$x = a \cdot e^{(-\zeta + \sqrt{\zeta^2-1})\omega_m t} + b \cdot e^{(-\zeta - \sqrt{\zeta^2-1})\omega_m t}$

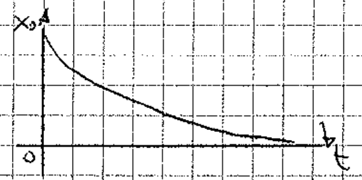
$\zeta > 1$ radici reali

$\zeta = 1$ LIMITE

$\zeta < 1$ radici complesse

- $\zeta > 1$: λ_1, λ_2 : 2 radici REALI DISTINTE < 0

se $\lambda < 0$ $\lim_{t \to \infty} e^{\lambda t} = 0$

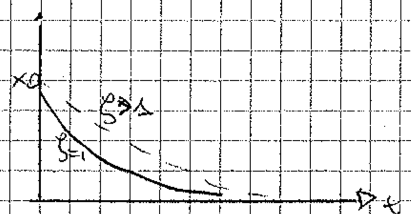


MOTO APERIODICO
↓
SISTEMA SOVRASMORZATO

- $\zeta = 1$ (SMORZAMENTO CRITICO)

$\lambda_1 = \lambda_2 = -\zeta\omega_m t = -\omega_m \cdot t$

$x = a e^{-\omega_m t} + bt e^{-\omega_m t} = (a + bt) e^{-\omega_m t}$



MOTO APERIODICO

- $\zeta < 1$

$\lambda_1, \lambda_2 = -\zeta\omega_m \pm \omega_m \sqrt{-1}\sqrt{1-\zeta^2} = -\zeta\omega_m \pm i\,\omega_m\sqrt{1-\zeta^2}$

$x = a e^{(-\zeta\omega_m + i\omega_m\sqrt{1-\zeta^2})t} + b e^{(-\zeta\omega_m - i\omega_m\sqrt{1-\zeta^2})t}$

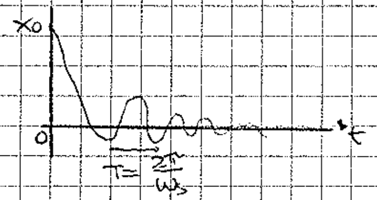
$\omega_m\sqrt{1-\zeta^2} = \omega_s$

PULSAZIONE NATURALE DEL SISTEMA SMORZATO

$\lambda_1, \lambda_2 = -\zeta\omega_m + i\omega_s$

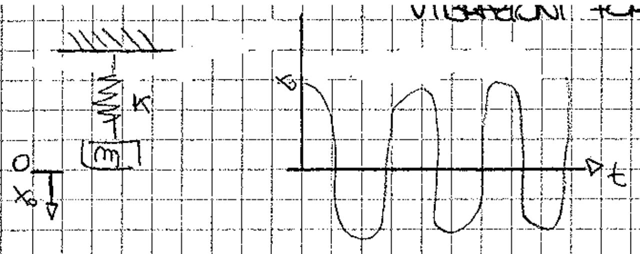
$x = a e^{(-\zeta\omega_m + i\omega_s)t} + b e^{(-\zeta\omega_m - i\omega_s)t} =$

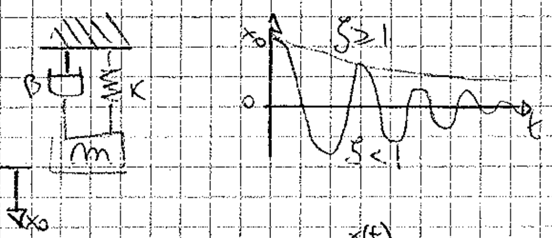
$= \underbrace{e^{-\zeta\omega_m t}}_{\to 0 \text{ per } t \to \infty} \underbrace{\left(a e^{i\omega_s t} + b e^{-i\omega_s t}\right)}_{x_0 \sin(\omega_s t + \varphi_0)}$

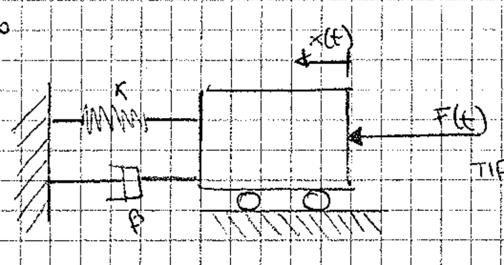


$x = x_0\, e^{-\zeta\omega_m t} \sin(\omega_s t + \varphi_0)$

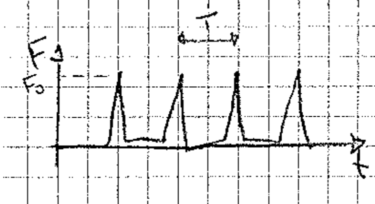
VIBRAZIONI FORZATE







FORZE DI TIPO PERIODICO



esempio: impulsi

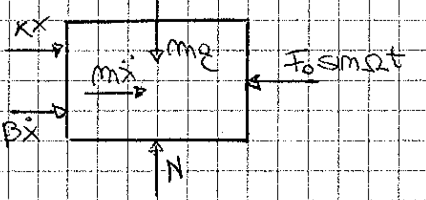
Se il sistema è LINEARE vale il PRINCIPIO DI SOVRAPPOSIZIONE DEGLI EFFETTI

una qualsiasi funzione periodica può essere scritta come somma di una costante più una serie di funzioni periodiche

$$F = F^* + \sum_n a_n \cos(n\omega t) + b_n \sin(n\omega t) \qquad \text{SERIE DI FOURIER}$$

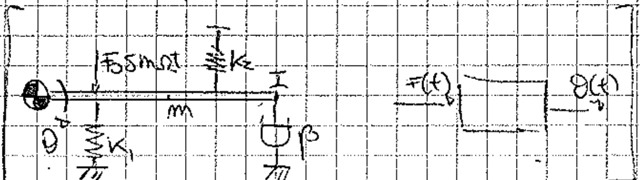
FORZANTE $\quad F(t) = F_0 \sin \Omega t$

ampiezza F_0
pulsazione Ω



$$m\ddot{x} + \beta\dot{x} + kx = F_0 \sin \Omega t$$

$$\ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin \Omega t$$



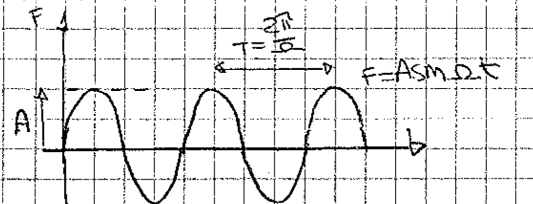
$\omega_n = f(k_1, k_2, m, I, a, b, c, d)$

$$\boxed{\ddot{\vartheta} + 2\zeta\omega_n\dot{\vartheta} + \omega_n^2\vartheta = A\sin\Omega}$$

equazione differenziale completa

ζ FATTORE DI SMORZAMENTO
ω_n^2 PULSAZIONE NATURALE
A AMPIEZZA

$$\underline{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A\sin\Omega t}$$



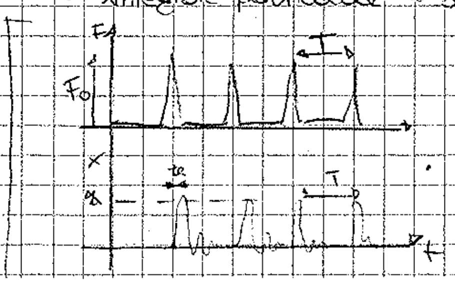
$x(t) = x_G + x_P \quad$ integrale generale + int. particolare

- $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

$x_G = X^* \sin(\omega_0 t + \varphi_0) e^{-\omega_n t} \quad \to 0 \text{ per } t \to \infty$

posso quindi trascurare x_G

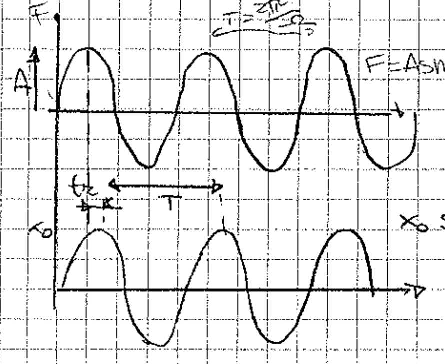
- integrale particolare → soluzione a regime quando si esaurisce il transitorio



il periodo di eccitazione e risposta è lo stesso

c'è un tempo di ritardo tra eccitazione e risposta

Se la FORZANTE è una sinusoide



INTEGRALE PARTICOLARE

$$x_p = x_0 \sin(\Omega t - \varphi)$$

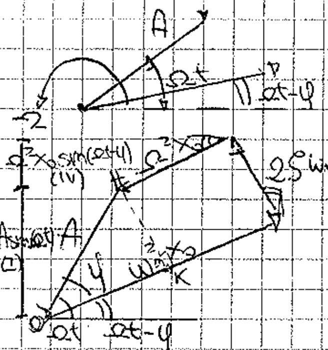
$$x_0 \sin(\Omega t - \varphi) = x_p$$

$$x = x_0 \sin(\Omega t - \varphi)$$

$$\dot{x} = x_0 \Omega \cos(\Omega t - \varphi)$$

$$\ddot{x} = -x_0 \Omega^2 \sin(\Omega t - \varphi)$$

$$\underbrace{-x_0 \Omega^2 \sin(\Omega t - \varphi)} + \underbrace{2\xi\omega_n \Omega x_0 \cos(\Omega t - \varphi)} + \underbrace{\omega_n^2 x_0 \sin(\Omega t - \varphi)} = \underbrace{A\sin\Omega t}$$



$A\sin\Omega t$
$x_0\sin(\Omega t-\varphi)$

$2\xi\omega_n\Omega x_0 \cos(\Omega t-\varphi)$ (III)

$\omega_n^2 x_0 \sin(\Omega t-\varphi)$ (II)

$$\begin{cases} OH^2 = OK^2 + HK^2 \\ tg\,\varphi = \frac{HK}{OK} \end{cases}$$

$$A^2 = x_0^2(\omega_n^2 - \Omega^2) + (2\xi\omega_n\Omega x_0)^2 \qquad x_0 = \frac{A}{\sqrt{(2\xi\omega_n\Omega)^2 + (\omega_n^2 - \Omega^2)^2}}$$

$$\varphi = tg^{-1}\frac{2\xi\omega_n\Omega x_0}{x_0(\omega_n^2 - \Omega^2)}$$

$$x_0 = \frac{A/\omega_n^2}{\sqrt{\left(2\xi\frac{\Omega}{\omega_n}\right)^2 + \left(1 + \frac{\Omega^2}{\omega_n^2}\right)^2}} \qquad \frac{x_0}{A/\omega_n^2} = \frac{1}{\sqrt{\left(2\xi\frac{\Omega}{\omega_n}\right)^2 + \left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}} = M$$

FATTORE DI AMPLIFICAZIONE

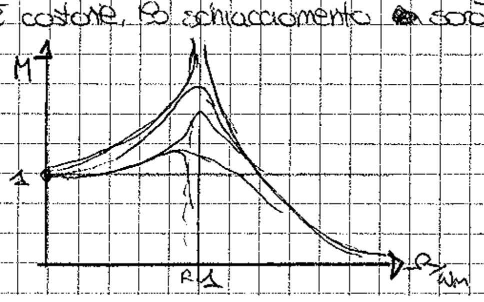
$$m\ddot{x} + \beta\dot{x} + kx = F_0 \sin\Omega t$$

$$\omega_n^2 = \frac{K}{m}$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = A\sin\Omega t = \frac{F_0}{m}\sin\Omega t$$

$$\frac{A}{\omega_n^2} = \frac{F_0}{m}\cdot\frac{m}{K} = \frac{F_0}{K}$$

Se F_0 è costante, lo schiacciamento sarà statico $x_{st} = \frac{F_0}{K}$

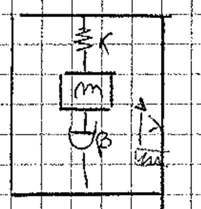


Se il sistema non è smorzato, ho questo andamento

Se aggiungo lo smorzamento, non tendono ad infinito, ma hanno dei massimi sopra 1

↳ l'ampiezza delle oscillazioni raggiunge un massimo aumentando (RISONANZA) e poi diminuisce

MISURA DELLE VIBRAZIONI



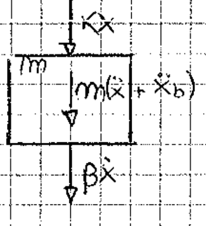
$X(t)$: posizione RELATIVA: MASSA / TELAIO

$X_b(t)$: posizione ASSOLUTA del TELAIO (CORPO VIBRANTE)

$x_b(t) = b \sin \Omega t$

$\dot{x}_b(t) = b\,\Omega \cos \Omega t$

$\ddot{x}_b = -b\,\Omega^2 \sin \Omega t$

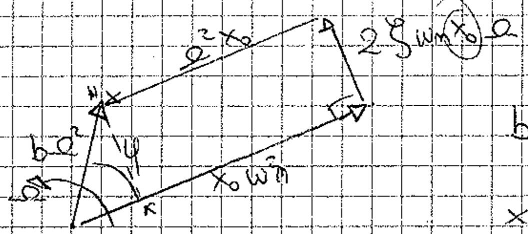


$m\ddot{x} + \beta\dot{x} + Kx + m\ddot{x}_b = 0$

$$\underline{m\ddot{x} + \beta\dot{x} + Kx = m b\,\Omega^2 \sin \Omega t}$$

$$\ddot{x} + \underbrace{\left(\frac{\beta}{m}\right)}_{2\zeta\omega_n}\dot{x} + \underbrace{\left(\frac{K}{m}\right)}_{\omega_n^2}x = b\,\Omega^2 \sin \Omega t$$

$x(t) = X_0 \sin(\Omega t - \varphi)$

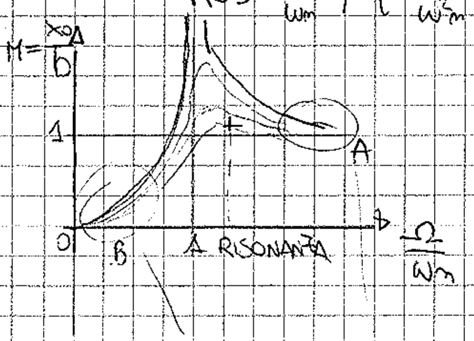


$$b\,\Omega^2 = \sqrt{OK^2 + HK^2} = \sqrt{X_0^2(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2 X_0^2}$$

$$X_0 = \frac{b\,\Omega^2}{\sqrt{(2\zeta\omega_n\Omega)^2 + (\omega_n^2 - \Omega^2)^2}}$$

$$\frac{X_0}{b} = \frac{\Omega^2/\omega_n^2}{\sqrt{\left(2\zeta\frac{\omega_n\Omega}{\omega_n}\right)^2 + \left(\frac{\omega_n^2 - \Omega^2}{\omega_n^2}\right)^2}}$$

$$\frac{X_0}{b} = \frac{\Omega^2/\omega_n^2}{\sqrt{\left(2\zeta\frac{\Omega}{\omega_n}\right)^2 + \left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}} = M$$



per $\zeta = 0$ $\quad M = \frac{X_0}{b} = \dfrac{\Omega^2/\omega_n^2}{1 - \frac{\Omega^2}{\omega_n^2}}$

Si individuano due zone di lavoro $\quad \Omega \gg \omega_n$, $\quad \Omega \ll \omega_n$

Ⓐ $\Omega \gg \omega_n \quad M = \frac{X_0}{b} \cong 1 \qquad X_0 \cong b \quad$ SISMOGRAFO

m : massa sismica

$\omega_n = \sqrt{\frac{K}{m}}$

Ⓑ $\Omega \ll \omega_n \quad M = \frac{X_0}{b} = \frac{\Omega^2}{\omega_n^2} \qquad X_0 = \frac{1}{\omega_n^2}(b\,\Omega^2)$

$b\,\Omega^2$: AMPIEZZA dell'ACCELERAZIONE

ACCELEROMETRO