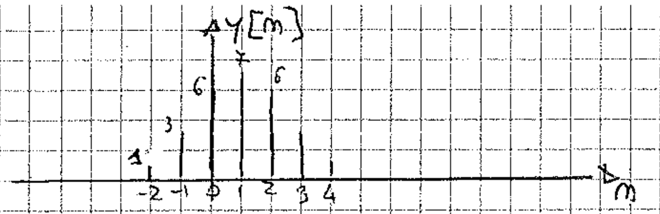


$$y[n] = \delta[n+2] + 3\delta[n+1] + 6\delta[n] + 7\delta[n-1] + 6\delta[n-2] + 3\delta[n-3] + \delta[n-4]$$

TRASFORMATA Z

$$\delta[n] \xrightarrow{Z} 1$$

$$x[n] * y[n] \xrightarrow{Z} X(z) \cdot Y(z)$$

$$\boxed{X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}}$$

$$\sum_{0}^{+\infty} a^{-n} = \frac{1}{1-a^{-1}} \quad |a|$$

- $X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$ $\qquad 1 \cdot z^0 = 1$
- $X(z) = \sum_{k=-\infty}^{+\infty} u[k]\, z^{-k} = \sum_{k=0}^{+\infty} z^{-k}$ $\qquad |z| > 1$

$$X(z) = \sum_n x[n]\, z^{-n}$$
$$Y(z) = \sum_k y[k]\, z^{-k}$$

$$z[n] = \sum x[p]\, y[n-p]$$
$$Z(z) = \sum x[p]\, y[n-p]\, z^{-n} = \underbrace{\sum x[n] z^{-n}}_{X(z)} \underbrace{\sum y[k] z^{-k}}_{Y(z)} = \sum_n \sum_k x[n] z^{-n} y[k] z^{-k} =$$
$$= \sum_{n,k} x[n]\, y[k]\, z^{-(k+n)} = \sum x[n]\, y[p-n]\, z^{-p}$$

$p = n + k$
$n = p - k$
$k = p - n$

$$x[n-N] = \sum_n x[n-N]\, z^{-n} = \sum_p x[p]\, z^{-(p+N)} = \sum_p x[p]\, z^{-p} z^{-N} = X(z)\, z^{-N}$$

$n - N = p$
$n = p + N$

$$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a} \qquad |a| < 1$$

$$\sum_{n=0}^{+\infty} a^{-n} = \frac{1}{1-a^{-1}} \qquad |a| > 1$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

$$X(z) = \sum_{n=0}^{0} \delta[n]\, z^{-n} = 1$$

$$\overbrace{y[n]}^{1} = \overbrace{x[n] * h[n]}^{2} \quad \downarrow ?$$
$$Y(z) = X(z)H(z)$$

Proviamo a dimostrarlo

1) $Y(z)=\sum_{n=-\infty}^{+\infty} y[n]\, z^{-n}$

2) $\sum_{n=-\infty}^{+\infty}\left(\sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]\right) z^{-n} = \sum_{k=-\infty}^{+\infty} x[k] \sum_{n=-\infty}^{+\infty} h[n-k]\, z^{-n}$ chiamo $n-k=p$, $n=k+p$

$= \sum_{k=-\infty}^{+\infty} x[k] \sum_{p=-\infty}^{+\infty} h[p]\, z^{-(k+p)} = \sum_{k=-\infty}^{+\infty} x[k] \sum_{p=-\infty}^{+\infty} h[p]\, z^{-p} z^{-k} =$

$= \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}}_{X(z)} \underbrace{\sum_{p=-\infty}^{+\infty} h[p]\, z^{-p}}_{H(z)}$

TRASFORMATA Z DEL GRADINO

• $u[n]$

$U(z)=\sum_{n=-\infty}^{+\infty} u[n]\, z^{-n} = \sum_{n=0}^{+\infty} z^{-n} = \frac{1}{1-z^{-1}} \qquad |z|>1$

• $u[-n]$

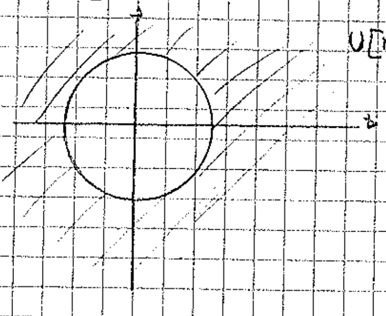
$U(z) = \sum_{n=0}^{-\infty} z^{-n} \overset{n=-n}{=} \sum_{n=0}^{+\infty} z^{n} = \frac{1}{1-z} \qquad |z|<1$

• $-u[-n-1]$

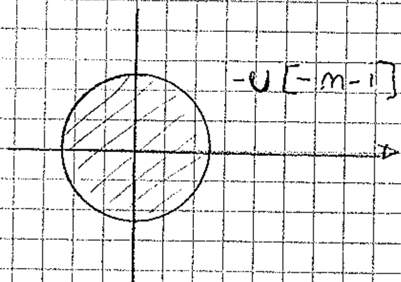
$U(z)=\sum_{n=-\infty}^{+\infty} -u[-n-1]\, z^{-n} = -\sum_{n=-1}^{-\infty} z^{-n} \overset{n=-n}{=} -\sum_{n=1}^{+\infty} z^{n} = -\left(\sum_{n=0}^{+\infty} z^{n} - 1\right) =$

$= -\frac{1}{1-z} + 1 = \frac{1+z-1}{z-1} = \frac{z}{z-1} \qquad |z|<1$

Le trasformate z di $u[n]$ e $-u[-n-1]$ sono uguali, però hanno regioni di convergenza diverse



$u[n]$



$-u[-n-1]$

$X(z)=\frac{N(z)}{D(z)}$

$u[n] \longrightarrow \frac{1}{1-x^{-1}}$

$\downarrow$

$\frac{1}{1-e^{-j2\pi f}}$

LE SEQUENZE CAUSALI SONO NULLE PER $n<0$

LE SEQUENZE ANTICAUSALI SONO NULLE PER $n>0$

$\sum_{m=0}^{N} \alpha_m\, \delta[n-m] \longrightarrow \sum_{m} \alpha_m z^{-m}$

1) $\sum_{0}^{+\infty} u[n] \longrightarrow X(z)=\frac{1}{1-z^{-1}} \qquad |z|>1$

2) $\sum_{n=0}^{-\infty} u[-n] \longrightarrow X(z)=\frac{1}{1-z} \qquad |z|<1$

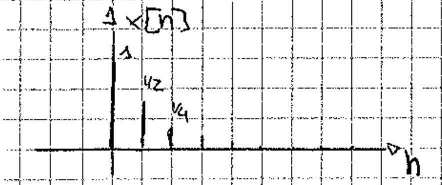
da 0 a $+\infty$

TUTTE LE SEQUENZE CAUSALI CONVERGONO ALL'~~INTERNO~~ ESTERNO DI UN CERCHIO

da 0 a $-\infty$

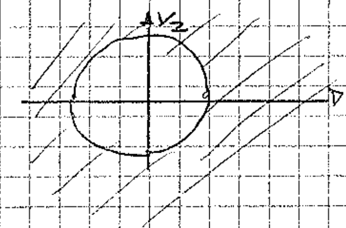
TUTTE LE SEQUENZE ANTICAUSALI CONVERGONO ~~AL ESTERNO~~ INTERNO DI UN CERCHIO

- $x[n] = \left(\frac{1}{2}\right)^n u[n]$ SEQUENZA CAUSALE



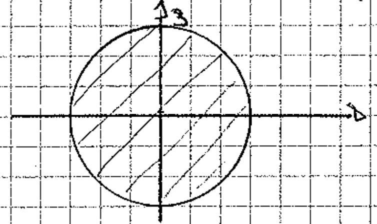
$$X(z) = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} =$$

$$= \frac{2z}{2z-1} \qquad \left|\frac{1}{2z}\right| < 1 \qquad |z| > \frac{1}{2}$$

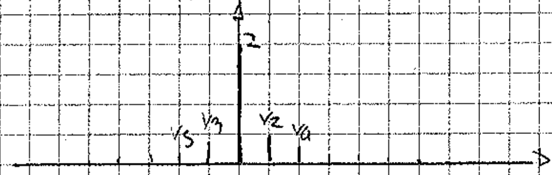


- $x[n] = 3^n u[-n]$ SEQUENZA ANTICAUSALE

$$X(z) = \sum_{n=0}^{-\infty} 3^n z^{-n} \underset{m=-n}{=} \sum_{m=0}^{+\infty} 3^{-m} z^{m} = \sum_{m=0}^{+\infty} \left(\frac{z}{3}\right)^m = \frac{1}{1-\frac{z}{3}} = \frac{3}{3-z} \qquad |z| < 3$$



- $x[n] = \left(\frac{1}{2}\right)^n u[n] + 3^n u[-n]$ $\longrightarrow$ $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z} = \frac{2z}{2z-1} + \frac{3}{3-z} =$



$$= \frac{6z - 2z^2 + 6z - 3}{(2z-1)(3-z)} = \frac{-2z^2 + 12z - 3}{(2z-1)(3-z)}$$

GLI ZERI DEL NUMERATORE SI CHIAMANO ZERI

GLI ZERI DEL DENOMINATORE SI CHIAMANO POLI $\longrightarrow$ L'ORDINE DI UN FILTRO È DATO DAL N° DEI POLI

UNA SEQUENZA BILATERALE CONVERGE IN UNA CORONA CIRCOLARE

- $x[n] = \alpha^n u[n]$ $\quad \alpha > 0$

$$X(z) = \sum_{n=0}^{+\infty} \alpha^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{\alpha}{z}\right)^n = \frac{1}{1-\frac{\alpha}{z}} = \frac{z}{z-\alpha} \qquad |z| > \alpha$$

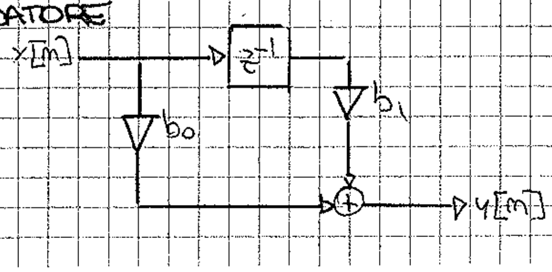
POLO: $z = \alpha$

- $x[n] = \alpha^n u[-n]$

$$X(z) = \sum_{n=0}^{-\infty} \alpha^n z^{-n} = \sum_{m=0}^{+\infty} (\alpha^{-1} z)^m = \frac{1}{1-\frac{z}{\alpha}} = \frac{\alpha}{\alpha - z} \qquad |z| < \alpha$$

POLO: $z = \alpha$

RITARDATORE



$$\begin{cases} x[n-T] = X(z) z^{-T} \\ x[n] \xrightarrow{z^{-1}} x[n-1] \end{cases}$$

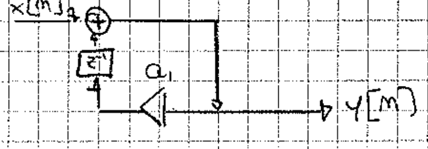
$$y[n] = b_1 x[n-1] + b_0 x[n]$$

$$Y(z) = b_1 X(z) z^{-1} + b_0 X(z) = X(z)(b_1 z^{-1} + b_0)$$

$$H(z) = \frac{Y(z)}{X(z)} = b_1 z^{-1} + b_0$$

$$h[n] = b_1 \delta[n-1] + b_0 \delta[n]$$

RETROAZIONE (FEEDBACK)



$$y[n] = x[n] + a_1 y[n-1]$$

$$Y(z) = X(z) + a_1 Y(z) z^{-1}$$

$$Y(z)(1 - a_1 z^{-1}) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a_1 z^{-1}}$$

$$h[n] = (a_1)^n u[n] \qquad |z| > a_1$$

I FILTRI

FILTRO A CASCATA → FIR = risposta all'impulso finita

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + \dots + b_N x[n-N] +$$
$$+ a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] + a_M y[n-M]$$

SE SONO PRESENTI ANCHE I TERMINI CON LA 'a' ⇒ IIR risposta all'impulso infinita

$$Y(z) = X(z)(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \dots + b_N z^{-N}) +$$
$$+ Y(z)(a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \dots + a_M z^{-M})$$

$$Y(z)(1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3} \dots - a_M z^{-M}) = X(z)(b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N})$$

$$\Downarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N}}{1 - a_1 z^{-1} - a_2 z^{-2} - \dots - a_M z^{-M}}$$

gli zeri di questo polinomio sono gli zeri del filtro

gli zeri del denominatore sono i poli del filtro e indicano l'ordine del filtro

1) $h[n]$

2) I/O $\quad y[n] = \sum_{0}^{N} b_k x[n-k] + \sum_{1}^{M} a_k y[n-k] =$

$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \dots + b_N x[n-N] +$$
$$a_1 y[n-1] + a_2 y[n-2] + \dots + a_M y[n-M]$$

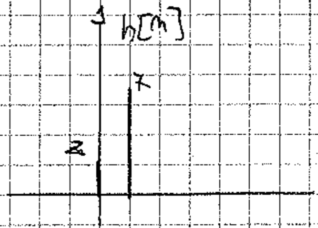
$$Y(z)(1 - a_1 z^{-1} - a_2 z^{-2} + \dots + a_M z^{-M}) = X(z)(b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N})$$

3) $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N}}{1 - a_1 z^{-1} - a_2 z^{-2} - \dots - a_M z^{-M}}$

Per progettare un filtro si utilizza la 3 → [H(z)]

FINITE IMPULSE RESPONSE

$H(z) = \frac{N(z)}{1}$



- $y[n] = 2x[n] + 7x[n-1]$ $\qquad H[z] = 2 + 7z^{-1}$

  $h[n] = 2\delta[n] + 7\delta[n-1]$

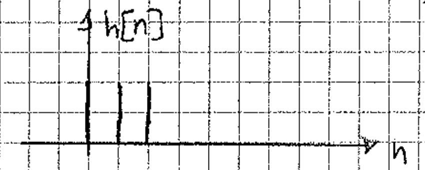
STABILITÀ : BIBO → bounded input bounded output

↳ se gli ingressi sono limitati anche le uscite sono limitate

$y[n] = \frac{1}{n} x[n]$ $\qquad |h[n]| < \infty$

il filtro FIR è sempre stabile, anche se non ha grandi prestazioni

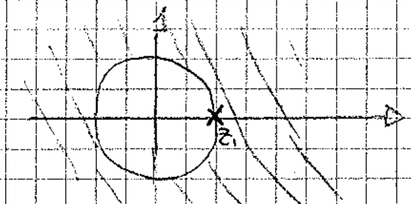
- $h[n] = u[n] - u[n-3]$



INFINITE IMPULSE RESPONSE

- $h[n] = u[n]$

  $H(z) = \frac{1}{1 - z^{-1}}$ $\qquad |z| > 1$



la regione di convergenza ci dice dove la trasformata z esiste

i poli ci dicono la stabilità del sistema

- $\frac{(1 - 2z^{-1})(1 - z^{-1})}{1 - \frac{1}{2}z^{-1}}$

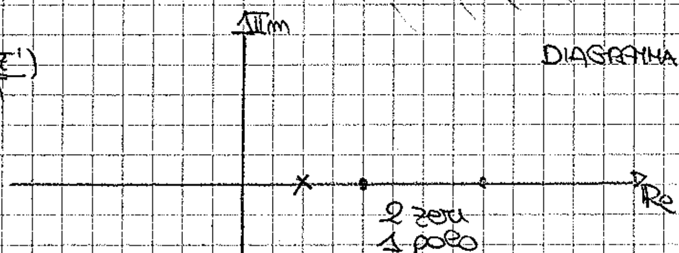


DIAGRAMMA ZERI-POLI

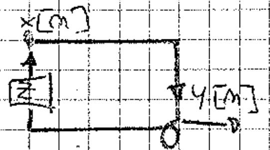
- $\frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1}}$

  $Y(z)(1 - z^{-1}) = X(z)$

  $Y(z) - Y(z)z^{-1} = X(z)$

  $y[n] - y[n-1] = x[n]$

  $y[n] = y[n-1] + x[n]$ $\qquad$ I/O



filtri IIR sono stabili se i poli sono all'interno del cerchio unitario

- I/O, $h[n]$, $H(z)$

  $h[n] = u[n] - u[n-3]$

  1) IIR anche se è sbagliato

  2) $H(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - z^{-1}} z^{-3}$ $\qquad |z| > 1$

$$H(z) = \frac{1-z^{-3}}{1-z^{-1}} = \frac{(1-z^{-1})(1+z+z^{-2})}{\cancel{1-z^{-1}}} \qquad \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2}$$

il denominatore è unitario, x ciò è un filtro FIR

$$y[n] = x[n] + x[n-1] + x[n-2]$$

- $y[n] = \frac{1}{2} y[n-1] + x[n]$

1) $\sum |h[n]| < \infty$

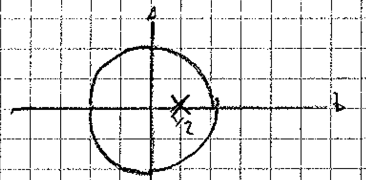
2) $H(z)$

$$h[n] = \frac{1}{2} h[n-1] + \delta[n]$$

$$Y(z)(1 - \tfrac{1}{2} z^{-1}) = X(z)$$ ho trasformato la y[n] nel dominio z

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$$ FILTRO IIR (il denominatore contiene un polo)

polo $z = \frac{1}{2}$ → il sistema è stabile

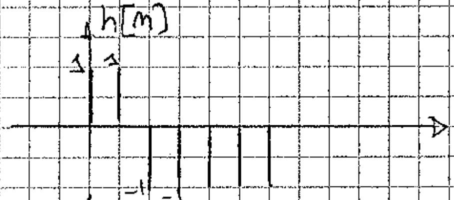


$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] \qquad |z| > \tfrac{1}{2}$$

Ogni trasformata z corrisponde a due sequenze: una causale e una anticausale

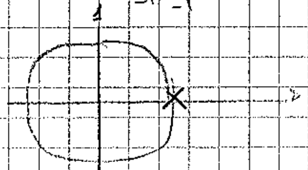
- $h[n] = u[n] - 2u[n-2]$

a) stabilità
b) $H(z)$
c) I/O relazione input/output



$$h[n] = \delta[n] + \delta[n-1] - u[n-2]$$

$$H(z) = 1 + z^{-1} - \frac{1}{1-z^{-1}} z^{-2}$$



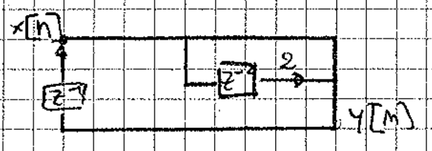
IL POLO SUL CERCHIO DI RAGGIO UNITARIO NON È STABILE

$$H(z) = \frac{(1+z^{-1})(1-z^{-1}) - z^{-2}}{1-z^{-1}} = \frac{1-z^{-2}-z^{-2}}{1-z^{-1}} = \frac{1-2z^{-2}}{1-z^{-1}} = \frac{Y(z)}{X(z)}$$

$$X(z)(1-2z^{-2}) = Y(z)(1-z^{-1})$$

$$y[n] - y[n-1] = x[n] - 2x[n-2]$$

$$y[n] = y[n-1] + x[n] - 2x[n-2]$$



- $h[n] = \delta[n] + 6\delta[n-1] + 3\delta[n-2]$

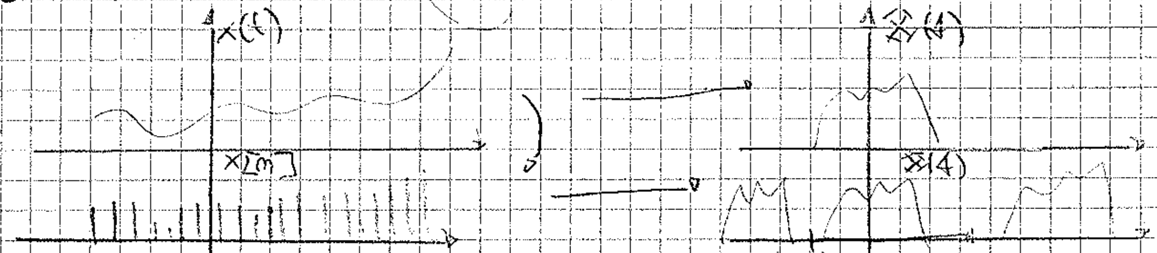
$$H(z)\Big|_{z=e^{j2\pi f}} \longrightarrow H(f)$$

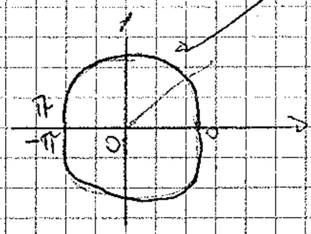
$$H(z) = 1 + 6z^{-1} + 3z^{-2}$$

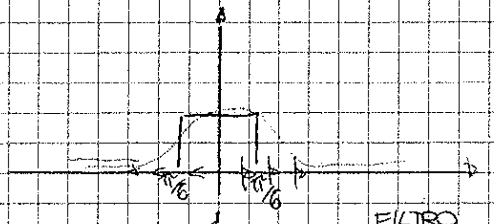
$$H(f) = 1 + 6e^{-j2\pi f} + 3e^{-j4\pi f} \qquad 1 + e^{j2\pi f} = 2e^{j\pi f}\left(\frac{e^{-j\pi f} + e^{j\pi f}}{2}\right)$$

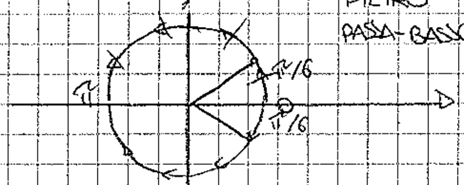
? Esercizi $\Rightarrow$?

$z = e^{j2\pi f}$ $\quad \mathcal{Z} \longrightarrow$ DTFT ?





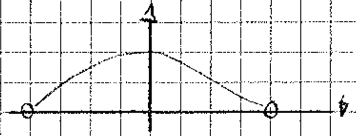


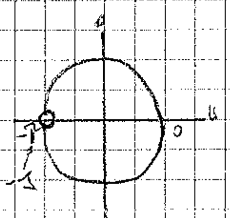


FILTRO PASSA-BASSO

Le frequenze basse sono vicine allo 0
le frequenze alte sono vicine a π

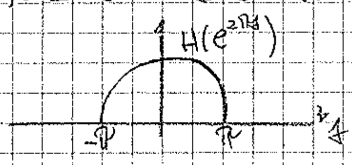
Il filtro passabasso migliore mette lo zero in 1 (π)





$$H(z) = z + 1 = 1 + z^{-1} \qquad H(f) = 1 + e^{-j2\pi f} = e^{-j\pi f}\left(e^{-j\pi f} + e^{j\pi f}\right)$$

$$h[n] = \delta[n] + \delta[n-1]$$

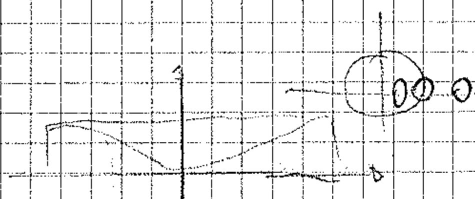


zero alle frequenze alte $\rightarrow$ FILTRO PASSA BASSO

zero alle frequenze basse $\rightarrow$ FILTRO PASSA ALTO

Posso ignorare gli zeri in zero e i poli in infinito

PASSA ALTO $\quad H(z) = z - 1 = 1 - z^{-1}$

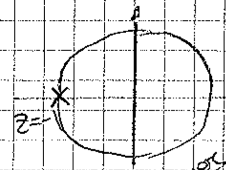


Lo zero posso metterlo in qualunque punto sul raggio $\rightarrow$ la distanza dal centro indica la velocità

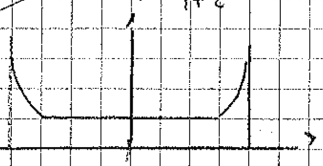
$$H(z) = \tfrac{1}{4}z - \tfrac{1}{3} = 1 - \tfrac{1}{3}z^{-1} \qquad \text{oppure} \qquad H(z) = 1 - 2z^{-1}$$

$$h[n] = \delta[n] - \tfrac{1}{3}\delta[n-1] \qquad h[n] = \delta[n] - 2\delta[n-1]$$

Quelli di primo ordine filtri FIR. Adesso i filtri IIR



$$H(z)=\frac{1}{1+z^{-1}}$$ FILTRO PASSA ALTO

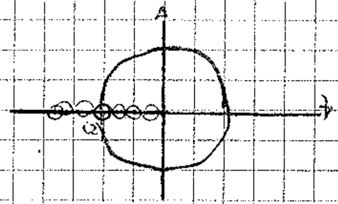


trasf di Fourier

I poli portano la $H(z)$ ad infinito → ordine

$$H(f)=\frac{1}{1+e^{-j2\pi f}}$$

* FIR passa basso con due zeri



↳ metto uno zero doppio in $z=-1$

Ma dato che la frequenza corrisponde ad un angolo posso metterle su tutto il raggio

$$H(z)=(1+z^{-1})^2=1+2z^{-1}+z^{-2}$$

$$h[n]=\delta[n]+2\delta[n-1]+\delta[n-2]$$

Aumentando il numero di zeri aumento l'ordine del filtro → si muove + velocemente

⇒ il coseno (uno zero) tende ad essere uno posto (infiniti zeri)

il filtro DISTORCE L'INFORMAZIONE NELLA BANDA = non tutte le frequenze hanno la stessa amplificazione

- $H(z)=(1+z^{-1})^3=1+z^{-3}+3z^{-2}+3z^{-1}$

$$h[n]=\delta[n]+3\delta[n-1]+3\delta[n-2]+\delta[n-3]$$

RIPASSO

$$y[n]=x[n]*h[n]$$

1) STABILITÀ

$$\sum|h[n]|<\infty$$

2) $H(z)=\frac{N(z)}{D(z)}$

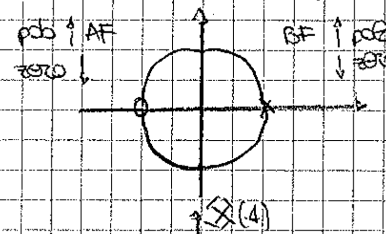
$H(z)=\frac{1}{1-pz^{-1}}$ $|z|>1$

$h[n]=p^n u[n]$

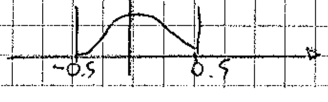
3) I/O $\frac{Y(z)}{X(z)}=\frac{1}{1-pz^{-1}}$

$Y(z)-pY(z)z^{-1}=X(z)$ $y[n]=py[n-1]+x[n]$

4) FIR $H(z)=\sum_{k=0}^{M}a_k z^{-k}$

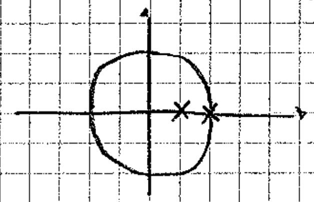


$$H(z)=\frac{1-a_0z^{-1}}{1-b_1z^{-1}}$$ a_0 → zero, b_1 → polo



$$H(z)=\frac{(1-a_0z^{-1})(1-a_1z^{-1})(1-a_2z^{-1})\dots(1-a_Nz^{-1})}{(1-b_1z^{-1})(1-b_2z^{-1})\dots(1-b_Mz^{-1})}$$

$$H(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$



$$\frac{A_1}{1-z^{-1}} + \frac{A_2}{1-0.5z^{-1}}$$

$$A_1 = \frac{H(z)}{1-z^{-1}}\bigg|_{z=1} = \frac{1}{1-0.5z^{-1}}\bigg|_{z=1} = 2$$

$$A_2 = \frac{1}{1-z^{-1}}\bigg|_{z=0.5} = \frac{1}{1-2} = -1$$

$$\frac{2}{1-z^{-1}} + \frac{-1}{1-0.5z^{-1}}$$

$$2u[n] - 1(0.5)^n u[n] = h[n]$$

$|z|>1$ $\quad$ $|z|>0.5$